Stratified mukai flops $\rightarrow$ derived equivalences.

Flops + derived equivalences

Cautis – Flops & about...

<u>Atiyah flops:</u> $Z = \{xy - wz\} \subset \mathbb{A}^4$

$$\mathcal{O}(-1,-1)_{\mathbb{P}^1\times\mathbb{P}^1} \cong \mathrm{Bl}_{\mathbb{P}^1}X \quad Y \quad \mathrm{Bl}_{\mathbb{P}^1}X' \quad \text{3d odp}$$

$$\begin{tikzcd} & Y \arrow[dl, "\pi"'] \arrow[dr, "\pi'"] & \\ X \arrow[dr] \arrow[rr, phantom, "\cong \mathcal{O}(-1)^{\oplus 2}_{\mathbb{P}^1} \cong"] & & X' \arrow[dl] \\ & Z & \end{tikzcd}$$

Small.

$$Y = X \times_Z X'$$

$$\mathbb{A}^4 \times \mathbb{P}^1$$
$$\cup$$
$$X = \left\{ xy - zw,\ \frac{x}{z} = \frac{s}{t} = \frac{w}{y} \right\}$$

$$X' = \left\{ \quad \frac{x}{w} \quad \frac{z}{y} \right\}$$

Bondal–orlov:

$$(\pi')_* \pi^* : D^b(X) \xrightarrow{\sim} D^b(X')$$

or $(\pi')_*(\pi(-) \otimes \mathcal{O}(k, l))$

$$D^b(X) = \langle \mathcal{O}, \mathcal{O}_{\mathbb{P}^1} \rangle \quad \text{or} \quad \langle \mathcal{O}, \mathcal{O}(1) \rangle$$

$$\mathrm{Ext}^k(\mathcal{O} \oplus \mathcal{O}(1), \mathcal{O} \oplus \mathcal{O}(1)) = 0 \quad \text{if } k > 0$$

euler:

$$0 \to \mathcal{O}(1)^{\oplus 2} \to \mathcal{O}(2)$$

is exact.

$k = G$

$$\Rightarrow D^b(X) \cong D^b(A\text{-mod}) \qquad \begin{tikzcd} \mathcal{O} \arrow[r, bend left, "x_0, y_0"] & \mathcal{O}(1) \arrow[l, bend left, "x_1, y_1"] \end{tikzcd}$$

$$F \longmapsto \mathrm{Hom}(\mathcal{O} \oplus \mathcal{O}(1), F)$$

$$x_0 y_1 y_0 = y_0 y_1 x_0$$

$$\rightsquigarrow \quad x = x_0 x_1$$
$$y = y_0 y_1$$
$$z = x_0 y_1$$
$$w = y_0 x_1$$

$$\Rightarrow xy = zw$$

flop = birational, $K_X \cong K_{X'}$

~~$\mathrm{Ext}(\mathcal{O}(0), \mathcal{O}(1)) = R\Gamma(\mathcal{O}(1)) \neq 0$~~

i.e.
$$\begin{tikzcd} & Y \arrow[dl, "\pi"'] \arrow[dr, "\pi'"] & \\ X \arrow[rr, leftrightarrow, dashed] & & X' \end{tikzcd}$$

s.t. $\pi^* K_X = (\pi')^* K_{X'}$

'K - equivalence'

Thm: true in 3d (Bridgeland)

<u>Thm: (Kaledin)</u> True for symplectic resolutions of ~~sing~~ symplectic singularities.

$\mathcal{O}_{\mathbb{P}^1} \in D^b(X)$ is 3-spherical. $X = (ab - cd) \subseteq \mathbb{A}^4$

$(\pi'_*)(\pi)^* = \mathrm{Flop}$ $\quad \mathrm{Flop} \circ \mathrm{Flop} \neq id_{D^b(X)}$ = Spherical twist

$\mathbb{C}^4/\mathbb{C}^*$ with wts
$$\begin{array}{cccc} +1 & +1 & -1, & -1 \\ x & y & s & t \end{array}$$

$a = xs$

$b = yt$

$c = xt$

$d = ys$

2 GIT quotients

$\mathbb{P}^1_{x,y} \subseteq X = (\mathbb{C}^4 \setminus \{x=y=0\})/\mathbb{C}^*$

$\mathbb{P}^1_{s,t} \subseteq X' = (\mathbb{C}^4 \setminus \{s=t=0\})/\mathbb{C}^*$

$= Z$

V 2 dim

L 1 dim

$$\underbrace{\mathrm{Hom}(V, L)}_{\cong\, \mathbb{C}^2} \oplus \underbrace{\mathrm{Hom}(L, V)}_{\cong\, \mathbb{C}^2} \Big/ \underbrace{GL(L)}_{\cong\, \mathbb{C}^*}$$

$$X = \mathrm{Tot}\left(\mathrm{Hom}(L,V) \longrightarrow \mathbb{P}V^\vee \ \text{(1d subspace of } V^\vee\text{)}\right)$$
$$= \{ V \twoheadrightarrow L,\ \alpha : L \to V \}$$

$$X' = \mathrm{Tot}\left(\mathrm{Hom}(V,L) \longrightarrow \mathbb{P}V \right)$$
$$= \{ \underset{M}{L} \subseteq V,\ \beta : V \to \underset{M}{L} \}$$

$$Y = \{ V \twoheadrightarrow L,\ M \subseteq V,\ \gamma : L \to M \}$$

$\mathbb{P}^1 \times \mathbb{P}^1$ Line bundle

$$\begin{tikzcd} & X \arrow[dr, "\text{forget } L"] & \\ Y \arrow[ur, "\text{forget } M"] \arrow[dr, "\text{forget } L"'] & & Z \\ & X' \arrow[ur, "\text{forget } M"'] & \end{tikzcd}$$

$$\{\mathrm{rk} \le 1\} = Z = (ab - cd) \subset \mathbb{A}^4 = \mathrm{Hom}(V,V)$$

<u>Now let $\dim V = n$</u>

dim $2n-1$ $\quad X = \underset{\text{VB}}{\text{rank } n}$ over $\mathbb{P}V^\vee$

Y is Line Bundle over $\mathbb{P}V^\vee \times \mathbb{P}V$

$X' = \qquad \mathbb{P}V$

$$X \cong \left(\mathcal{O}(-1)^{\oplus n} \text{ over } \mathbb{P}^{n-1}\right) \cong X'$$

$$Y \cong \mathcal{O}(-1,-1) \text{ over } \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$$

$$\begin{tikzcd} & Y \arrow[dl] \arrow[dr] & \\ X \arrow[dr, dash] & & X' \arrow[dl] \\ & Z & \end{tikzcd}$$

$Z \leftarrow \{\text{rank} \le 1\} \subseteq \mathrm{Hom}(V,V)$.

$D^b(X) \xrightarrow{\sim} D^b(X')$ Still have $Y = X \times_Z X'$

Mukai flops

hyperkahler / alg symplectic.

$$T^\vee(\mathbb{P}V^\vee) \xleftarrow{\text{flop}} - - T^\vee(\mathbb{P}V)$$

$q(\alpha,\beta)$ moment map

$$\underset{\beta}{\mathrm{Hom}(V,L)} \oplus \underset{\alpha}{\mathrm{Hom}(L,V)} \supseteq \{\langle\alpha|\beta\rangle = 0\} =: Q$$

quadratic cone

$$\{\langle\alpha|\beta\rangle = 0\} = \{L \xrightarrow{\alpha} V \xrightarrow{\beta} L\}$$

Nakajima quiver variety

① — ⓝ

$$Q \setminus \{\beta = 0\} / \mathbb{C}^\times \subset X$$
$$\| $$
$$T^*(\mathbb{P}V^\vee)$$

$$T^*\mathbb{P}^n \hookrightarrow \mathcal{O}(-1)^n \to \mathcal{O}$$
$$X \xrightarrow{q} \mathbb{C}$$

$$Q \setminus \{\alpha = 0\} / \mathbb{C}^\times \subset X'$$
$$\|$$
$$T^*(\mathbb{P}V)$$

$$\begin{tikzcd}[column sep=small] & L \arrow[dr] & \\ V \arrow[ur] \arrow[dr] & 0 & V \\ & M \arrow[ur] & \end{tikzcd} \hookrightarrow \begin{tikzcd}[column sep=small] & L \arrow[dl, shift left] \arrow[dd] & \\ V \arrow[ur, shift left] \arrow[dr, shift left] & 0 & V \arrow[ul, shift left] \arrow[dl, shift left] \\ & M \arrow[ul, shift left] \arrow[ur, shift left] & \end{tikzcd}$$

$$T^\vee(\mathbb{P}V^\vee) = \{V \twoheadrightarrow L,\ \alpha : L \to V,\ \underset{\text{is zero}}{L \to V \to L}\}$$

$$T^\vee(\mathbb{P}V) = \{M \subseteq V,\ \beta : V \to M,\ M \subseteq \ker\beta\}$$

i.e. $M \to V \to M$ is zero

both map to $\left\{ \begin{array}{l} rk \le 1 \\ \text{square to } 0 \end{array} \right\} \subseteq \{ rk \le 1 \} \subseteq \mathrm{Hom}(V, V) = \mathfrak{gl}(V)$

$\underbrace{}_{\text{minimal nilpotent closure.}}$ //

$Y = \{ V \twoheadrightarrow L,\ M \subseteq V,\ \varphi : L \to M$

$M \subseteq H$ ie. $M \hookrightarrow V \twoheadrightarrow L$ is zero $\}$

$H = \ker V \twoheadrightarrow L$

$= \mathrm{Tot}\big(\mathrm{Hom}(L, M)\big)$ $\to$ $\mathrm{Hom}(L, V)_{\mathbb{P}V^\vee}$, $\to$ $\mathrm{Hom}(V, M)_{\mathbb{P}V}$

$\downarrow$

$Fl(1, n-1, V)$

This is not the fibre product

$$T^*\mathbb{P}V^\vee \times_{\mathrm{sing}} T^*\mathbb{P}V$$

e.g. $Z = (ab - cd) \subset \mathbb{C}^4$

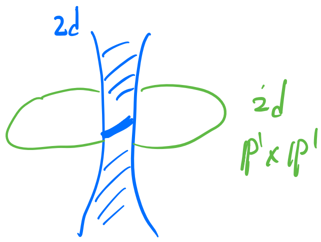
$\cup$

$X \quad X' \to (ab - cd,\ a+b) \subset \mathbb{C}^4$

$\cup \quad \cup \qquad\qquad \shortparallel$

$T^*\mathbb{P}^1 \quad T^*\mathbb{P}^1 \qquad \hat{Z}$

$T^*\mathbb{P}^1 \times_{\hat{Z}} T^*\mathbb{P}^1 \supset \mathbb{P}^1 \times \mathbb{P}^1$



Blue component is the line bundle on $Fl(1, n-1, V)$ $\neq \mathbb{P}^1$ i.e. Y

$\mathfrak{gl}(V) \supseteq X_0 = \{ rk \le 1 \} \supset Y_0 = \{ \xi \text{ rank} \le 1,\ \xi^2 = 0 \}$

Sing at 0 only — nilp orb closure

$X_+ = \{ \underset{1d}{L} \subset V,\ V \xrightarrow{\alpha} L \}$ is a VB on $\mathbb{P}V$

fibre is $Hom(V, L) = \mathcal{O}(-1)^{\oplus n}$

$\supset Y_+ = \{ L \subset \ker \alpha \} = T^*\mathbb{P}V$
VB on $\mathbb{P}V$
fibre is $Hom(V/L, L)$

or $X_- = \{ \overset{H \cap}{V} \twoheadrightarrow \underset{1d}{Q},\ Q \xrightarrow{\beta} V \}$ VB on $\mathbb{P}V^\vee$,

elt of $\mathbb{P}V^\vee$

fibre is $Hom(Q, V) \cong \mathcal{O}(-1)^{\oplus n}$.

$\supset Y_- = \{ \operatorname{im}\beta \subset H \}$, fibre is $Hom(Q, H) = T^*\mathbb{P}V$

or $\tilde{X} = \{ L \subset V,\ V \twoheadrightarrow Q,\ Q \to L \}$ line bundle $L \otimes Q^\vee$ on $\mathbb{P}V \otimes \mathbb{P}V^\vee$.

$\mathcal{O}(-1,-1)$

$\supseteq \tilde{Y} = \{ L \hookrightarrow V \twoheadrightarrow Q \text{ is zero} \}$
i.e. $L \subset H$ line bundle on $Fl(1, n-1, V)$
$\wr$ $\mathbb{P}V \times \mathbb{P}V^\vee$

$$\begin{tikzcd} & \tilde{X} \arrow[dl, "\pi_+"', "\text{Blowup}"] \arrow[dr, "\pi_-", "\text{Blowup}"'] & \\ X_+ \arrow[dr] & & X_- \arrow[dl] \\ & X_0 & \end{tikzcd}$$

$X_{\pm}$ are GIT quots of

$Hom(L, V) \oplus Hom(V, L) /\!/ GL(L)$

i.e. $\mathbb{C}^{2n} / \mathbb{C}^*_{1,\dots,1,-1,\dots,-1}$

Thm: (BO) $D^b(X_+) \longrightarrow D^b(X_-)$

via $(\pi_-)_*(\pi_+)^*$.

$$\begin{tikzcd} & \tilde{Y} \arrow[dl] \arrow[dr] & \\ Y_+ & & Y_- \end{tikzcd}$$

doesn't give derived equivalence.

$$\begin{tikzcd} & \tilde{Y} \subset \mathbb{P}V \times \mathbb{P}V^\vee \arrow[dl] \arrow[dr] & \\ Y_+ \arrow[dr] & & Y_- \arrow[dl] \\ & Y_0 & \end{tikzcd}$$

flag

this gives derived equivalence.

Grassmanian flops

$$X_0 = \{ \operatorname{rk} \leq k \}$$

Singular along

$$\operatorname{rk} \leq k-1$$

$$X_+ = \{ \underset{\underset{Gr(k,V)}{\cap}}{\overset{k\text{-dim}}{S}} \subset V,\ V \xrightarrow{\alpha} S \}$$

is a VB on $Gr(k, V)$

fibre is $Hom(V, S) \cong S^{\oplus n}$

taut. subbundle

$$X_- = \{ H \cap \underset{\underset{Gr(V,k)}{\cap}}{V} \twoheadrightarrow \overset{k\text{-dim}}{Q},\ Q \xrightarrow{\beta} V \}$$

is a VB on $Gr(V, k)$

fibre $Hom(Q, V)$

$X_\pm$ are GIT quots of $Hom(S,V) \oplus Hom(V,S) / GL(S)$

Thm (S-D): $D^b(X_+) \xrightarrow{\sim} D^b(X_-)$

Ballard-...

$$\tilde{X} = \{ S \subset V,\ V \twoheadrightarrow Q,\ Q \longrightarrow S \}$$

$$\begin{tikzcd} & \tilde{X} \arrow[dl] \arrow[dr] \arrow[d, phantom, "\vee"] & \\ X_+ \arrow[dr] & & X_- \arrow[dl] \\ & X_0 & \end{tikzcd}$$

$Y_0 = \{ \xi \text{ rank} \le k, \ \xi^2 = 0 \}$

nilpotent orbit closure

$Y_+ = \{ S \subset \ker \alpha \}$ VB on $Gr(k, V)$

fibre is $Hom(V/S, S)$

$= T^\vee Gr(k, V)$

$Y_- = T^\vee Gr(V, k)$ "Stratified Mukai flop"

$\tilde{Y} = \{ S \hookrightarrow V \to Q$ is zero, $Q \xrightarrow{\gamma} S \} \subset \tilde{X}$

VB over $Fl(k, n-k, V)$

$\cup$

$Gr(k, V) \times Gr(V, k)$

<u>Thm</u> (Cautis–Kamnitzer–Licata)

$\exists$ reflexive sheaf on $Y_+ \times_{Y_0} Y_-$ giving

$$D^b(Y_+) \xrightarrow{\sim} D^b(Y_-)$$

$$\begin{tikzcd} X \arrow[r, squiggly] \arrow[d, squiggly] & Aut(X) \arrow[d, hook] & \ni f \arrow[d, mapsto] \\ D^b(X) \arrow[r, squiggly] & Aut(D^b(X)) & \ni f_* \end{tikzcd}$$

$\cup\!|$

$\mathbb{Z}[1], Pic(X)$

$$\mathbb{Z} \times (Aut(X) \ltimes Pic(X)) \subseteq Aut(D^b(X))$$

<u>Thm</u>: (Bondal–Orlov)

This is an equality if ω_X is (anti)ample.

$\underline{Q}$: What if $\omega_X \cong \mathcal{O}_X$?

Spherical objects: (Siedel – Thomas)

$E \in D^b(X)$ is spherical

- $E \otimes \omega_X \cong E$
- $\mathrm{Hom}_{D^b(X)}(E, E[n]) = \begin{cases} \mathbb{C} & \text{if } n = 0, \dim X \\ 0 & \text{otherwise} \end{cases}$

$\left(\Rightarrow \mathrm{Hom}^{\cdot}(E,E) \cong H^{\cdot}(S^{\dim X}, \mathbb{C}) \right)$

Ex:
- C curve, $x \in C$, $\mathcal{O}_x$ is spherical
- $\mathrm{Tot}(\mathcal{O}(-1)_{\mathbb{P}^n}^{\oplus n+1}) \longrightarrow \mathcal{O}_{\mathbb{P}^n}$ is spherical

  (koszul res $s \in \mathcal{O}(1)^{\oplus n+1}$)
- S surface, $C \subseteq S$ -2-curve $\quad \mathcal{O}_C$ is spherical.

  $C \cong \mathbb{P}^1$
- X is strictly CY (i.e. $H^i(X, \mathcal{O}_X) = 0$ unless $i = 0, \dim X$)

  $\Rightarrow$ every line bundle is spherical.

Def^n: $T_E(M) := \mathrm{Cone}\left(R\mathrm{Hom}(E,M) \otimes_{\mathbb{C}} E \xrightarrow{ev} M\right)$

$M \in D^b(X)$

The spherical twist around E

Thm: T_E is an autoeq

Ex: $\cdot\ T_E(E) = \mathrm{Cone}\left(E \oplus E[-n] \longrightarrow E\right) \cong E[1-n]$

$\cdot$ If $R\mathrm{Hom}(E,M) = 0$, $\quad T_E(M) \cong M$.

$(X, \omega) \qquad\qquad X^\vee$

$D\mathrm{Fuk}(X,\omega) \simeq D^b(X^\vee)$

objs:

Lagrangians in $X \quad \rightsquigarrow \quad$ Dehn twist

1. $\mathcal{O}_x$ is spherical $\quad T_{\mathcal{O}_x} \simeq - \otimes \mathcal{O}_C(-x)$

(suppose) $f_*(- \otimes L)[p] \simeq T_E(-)$

$f_*(M \otimes L)[p] \simeq M \quad$ if $RHom(E, M) = 0$

$\Rightarrow p = 0$

$f_*(E \otimes L) \simeq E[1 - \dim X] \quad$ apply to E

$\dim X > 1$

$X_\pm = Tot(\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2})$

$$\begin{tikzcd}
 & Tot(\mathcal{O}(-1,-1)) \arrow[dl, "p"', "Bl_{\mathbb{P}^1}"] \arrow[dr, "q", "Bl_{\mathbb{P}^1}"'] & \\
X_- & & X_+
\end{tikzcd}$$

$\Phi := q_* p^* p_* q^* \in Aut(D^b(X_+))$

(<u>Theorem:</u> (Segal) Every autoeq is a spherical twist.)

$T_E^{-1}(M) \longrightarrow M \longrightarrow RHom(M, E)^\vee \otimes E$

Thm: $\Phi \simeq T^{-1}_{\mathcal{O}_{\mathbb{P}^1}(-1)}$

Pf: $\mathcal{O} \oplus \mathcal{O}(-1)$

$\Phi(\mathcal{O}) \simeq \mathcal{O}$

$$\begin{aligned}\Phi(\mathcal{O}(-1)) &= q_* p^* p_*(\mathcal{O}(0,-1)) \\ &\simeq q_* p^*(\mathcal{O}(1)) \\ &= q_*(\mathcal{O}(1,0)) \\ &= \mathcal{I}_{\mathbb{P}^1}(-1)\end{aligned}$$

- $R\mathrm{Hom}(\mathcal{O}, \mathcal{O}_{\mathbb{P}^1}(-1)) \simeq 0 \Rightarrow T^{-1}_{\mathcal{O}_{\mathbb{P}^1}(-1)}(\mathcal{O}) = \mathcal{O}$
- $R\mathrm{Hom}(\mathcal{O}(-1), \mathcal{O}_{\mathbb{P}^1}(-1)) \simeq \mathbb{C} \Rightarrow T^{-1}_{\mathcal{O}_{\mathbb{P}^1}(-1)}(\mathcal{O}(-1)) \simeq \mathcal{I}_{\mathbb{P}^1}(-1)$

$X_- = \mathrm{Tot}\left(\mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n+1}\right)$

Thm: (Addington–Donovsen–Meachan?)

$$q_* p^* p_* q^* \simeq T^{-1}_{\mathcal{O}_{\mathbb{P}^n}(-1)} \circ \cdots \circ T^{-1}_{\mathcal{O}_{\mathbb{P}^n}(-n)}$$

$\mathbb{P}^n$- object $\quad \mathrm{Hom}^\cdot(E,E) \simeq H^\cdot(\mathbb{P}^n, \mathbb{C})$

$$\wr\!\wr$$

$$\mathbb{C}[t]/t^{n+1} \quad \deg t = 2$$

P_E a new autoeq. (Huybrechts – Thomas)

P is a $\mathbb{P}^n$- obj $\quad t: P \to P[2]$

$$\left\{ \mathrm{RHom}(P,M)[-2] \otimes P \xrightarrow{t \otimes id - id \otimes t} \mathrm{RHom}(P,M) \otimes P \to M \right\}$$

$$\| $$

$$P_P(M)$$

if $n=1$, P is 2-spherical $\Rightarrow T_P^2 \simeq P_P$

$$D(\mathbb{C}[t])^c \xrightarrow[\underset{\mathbb{C}[t]}{-\otimes^L} P]{} D^b(X)$$

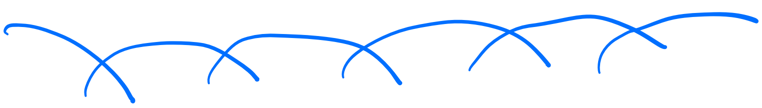
$\deg t = 2$

Braiding: $E, F \in D^b(X)$, spherical object

$$\mathrm{RHom}(E,F) = 0 \Rightarrow T_E T_F \simeq T_F T_E$$

$$\mathrm{RHom}(E,F) = \mathbb{C} \Rightarrow T_E T_F T_E = T_F T_E T_F$$

Ex: S surface $C_1, C_2 \simeq \mathbb{P}^1$, -2-curves s.t. $C_1 \cap C_2 = pt.$



Geometric categorical action of $\mathfrak{sl}_2$ on $T^*Gr(k,n)$

1. Reps of $\mathfrak{sl}_2$

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

n = highest weight

$$\underset{H}{\circlearrowleft} V_{-n} \overset{E}{\underset{F}{\rightleftarrows}} V_{-n+2} \quad \cdots\cdots \quad \underset{H}{\circlearrowleft} V_{n-2} \overset{E}{\underset{F}{\rightleftarrows}} V_n$$

Weyl element $T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = e^{E} e^{-F} e^{E} \in SL_2$

$$THT^{-1} = H$$

$$V_\lambda \xrightarrow{\sim} V_{-\lambda}$$

$$T = \frac{F^{\lambda}}{\lambda!} - \frac{F^{\lambda+1}E}{(\lambda+1)!} + \frac{F^{\lambda+2}E^{2}}{(\lambda+2)!\,2!} \cdots$$

2. Nakajima geo realisation

$$\begin{tikzcd} \bullet\, \mathbb{C}^n \arrow[d, shift left, "j"] \\ \bullet\, \mathbb{C}^k \arrow[u, shift left, "i"] \end{tikzcd} \qquad \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n) \oplus \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^k) \qquad G = GL_k$$

$$(i\ j) \longmapsto ji \qquad g.(i\ j) = (ig^{-1}, gj)$$

$$\mathcal{M}_\theta(k,n) := \mu^{-1}(0) /\!\!/_{\chi_\theta} G \qquad \chi_\theta : G \to \mathbb{C}^\times, \quad g \mapsto \det^{-\theta}$$

- $\theta = 0$, $\mathcal{M}_\theta = \mathrm{Spec}\,[\mu^{-1}(0)]^G \qquad (i\ j) \mapsto ij$

  $\simeq \overline{B(k)} := \{ x \in \mathrm{End}(\mathbb{C}^n) \mid x^2 = 0,\ rk(x) \leq k \}$

- $\theta > 0$, $\mathcal{M}_\theta = \mathrm{Proj} \bigoplus_{n \geq 0} \mathbb{C}[\mu^{-1}(0)]^{G, \chi_\theta} \qquad \lambda : \mathbb{C}^\times \hookrightarrow G$

By Mumford's Criterion.

$(i\ j)$ is semi-stable $\iff$ i is injective.

$$M_\theta = \{ i : \mathbb{C}^k \hookrightarrow \mathbb{C}^n,\ X \in \overline{B(k)} \}$$

$$\cong T^* Gr(k, n) \simeq Hom(\mathbb{C}^n/V, V)$$

- $\theta < 0$, $M_\theta \simeq \{ j : \mathbb{C}^n \twoheadrightarrow \mathbb{C}^k,\ B \in \overline{B(k)} \}$

$$\simeq T^* Gr(n-k, n)$$

Steinberg var
$$Z(k_1, k_2) = \left\{ (V_1, V_2, X) \;\middle|\; \begin{matrix} \operatorname{Im} X \\ \cap \\ V_i \\ \cup \\ \ker X \end{matrix} \right\} \left(= T^*Gr(k_1,n) \underset{B(k_1+k_2,n)}{\times} T^*Gr(k_2,n) \right)$$

$$\cap$$

$$T^*Gr(k_1, n) \times T^*Gr(k_2, n)$$

Hecke Correspondence.

$$B_k = \{ V_1 \xhookrightarrow{\text{codim } 1} V_2 \}$$

$$\subset T^*Gr(k,n) \times T^*Gr(k+1, n) \supset Z(k, k+1)$$

Nakajima

$$\hat{U(sl_2)} \xrightarrow{\text{alg hom}} \bigoplus_{k_1, k_2} H^{BM}_{top}(Z(k_1, k_2)) \curvearrowright \bigoplus_k H^{BM}_{top}(\pi^{-1}(0))$$

$\pi^{-1}(0) = Gr(k, n)$

$$(E)_k \longmapsto [B_k]$$

$$(F)_k \longmapsto [B_k^t]$$

Thm ✓ is irred rep of highest weight n.

Moreover: H^{BM}_{top} = weight space for $\lambda = n - 2k$

$$Y(\lambda) := T^* Gr(k, n)$$

$$\mathcal{E}^{(r)}_\lambda \in D(Y(\lambda) \times Y(\lambda+2r))$$

$$\mathcal{F}^{(r)}_\lambda \in D(Y(\lambda) \times Y(\lambda-2r))$$

$$T := \mathrm{Cone}\left(\mathcal{F}^{(r)}_\lambda \leftarrow \mathcal{F}^{(\lambda+1)}_{\lambda+2r} * \mathcal{E}^{(\lambda)}_\lambda[-1] \leftarrow \mathcal{F}^{(\lambda+2)}_{\lambda+4r} * \mathcal{E}^{(2)}_\lambda[-2] \leftarrow \cdots\right)$$

$$(\text{CKL}) \quad \Phi_T : D(Y(\lambda)) \xrightarrow{\sim} D(Y(-\lambda))$$

$$\mathcal{E}^{(s)}_\lambda := \mathcal{O}_{W^s_\lambda} \otimes \det(\mathbb{C}^n/V_1)^{-s} \otimes \det(V')^s$$

$\dim V' = k - s$

$$W^s_\lambda = \{(V_1, V', X) \mid \mathrm{Im}(X) \hookrightarrow V' \hookrightarrow V_1 \hookrightarrow \ker(X)\}$$

(V_1, V') in $Y(\lambda) \times Y(\lambda+2s)$

$$\mathcal{F}^{(\lambda+s)}_{\lambda+2s} := \mathcal{O}_{W^{\lambda+s}_{\lambda+2s}} \otimes \det(V_2/V')^{s}$$

$$\mathcal{F}^{(\lambda+s)}_{\lambda+2s} * \mathcal{E}^{(s)}_{\lambda}$$

$$\begin{tikzcd} Y(\lambda) & Y(\lambda) \times Y(\lambda+2s) \times Y(-\lambda) \arrow[l, "p_{12}"'] \arrow[d, "p_{23}"] \arrow[r, "p_{13}"] & {} \\ & {} & \end{tikzcd}$$

$$= p_{13,*}\Big(p_{12}^{*}\mathcal{O}_{W^{\lambda+s}_{\lambda+2s}} \otimes p_{23}^{*}\mathcal{O}_{W^{\lambda+s}_{\lambda+2s}} \otimes \det(\mathbb{C}^n/V_1)^{-s} \otimes \det(V')^{s} \otimes \det(V_2/V')^{s}\Big)$$

$$Z_s' := p_{12}^{-1}(W^{s}_{\lambda}) \cap p_{23}^{-1}(W^{\lambda+s}_{\lambda+2s}) \xrightarrow{\text{forget } V'} Z_s = \left\{ (V_1, V_2, S) \;\middle|\; \begin{array}{l} \mathrm{rk}\, X \leq k-s, \\ \dim(V_1 \cap V_2) \geq k-s \end{array} \right\}$$

$$= \{ \mathrm{Im}\, X \hookrightarrow V' \rightrightarrows \begin{smallmatrix} V_1 \\ V_2 \end{smallmatrix} \hookrightarrow \ker X \}$$

Component of $Y(\lambda) \times_{B(k)} Y(-\lambda)$

$Z_s^{o} \subset Z_s$ determined by

$\dim(V_1 \cap V_2) - \mathrm{rk}(X) \leq 1$,

(Cautis)

1. $p_{13,*}\mathcal{O}_{Z_s'} \cong j_* i_* \mathcal{O}_{Z_s^{o}}$

$$Z_s^{o} \xhookrightarrow{i} Y(\lambda) \times_{B(k)} Y(-\lambda) \xhookrightarrow{j} Y(\lambda) \times Y(-\lambda)$$

2. Z_s^{o} only intersect Z^{o}_{s-1} and Z^{o}_{s+1}

$D_s^{-} := Z_s^{o} \cap Z_s^{o}$, $D_s^{+} := Z_s^{o} \cap Z_{s+1}^{o}$

$$\mathcal{O}_{Z_s^\circ}([D_s^+] - [D_s^-]) \simeq \det(\mathbb{C}^n/V_1)^{-1} \otimes \det(V_2)$$

Glue line bundle $\mathcal{O}_{Z_s^\circ} \otimes \det(\mathbb{C}^n/V_1)^{-s} \otimes \det(V_2)^s$ on Z_s°

to a line bundle on $\bigsqcup_s Z_s^\circ$

(Cautis) $j_* i_* L \simeq T$

Nakajima's quiver varieties & Kac-Moody actions with a view toward/from symplectic resolution theory

Main ref: Lectures on Nakajima's quiver varieties by Victor Ginzburg. (And the references therein)

What do we do:

From Wei's talk, there were 3 things.

1) View things as special cases of Nakajima's quiver varieties, then apply Nakajima's results.

2) Categorify (CKL)

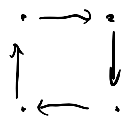
3) Do geometry? (C)

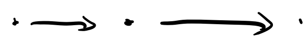
In this talk, we focus on 1), with emphasis on the symp resolution point of view.

More precisely, we are going to define general Nakajima's quiver varieties and study their (symplectic) geometric properties. Examples includes:
- Hilb ABC
- $\mathbb{C}^2/\Gamma$ $\quad \circ\!\longrightarrow - \cdots$
- [illegible]

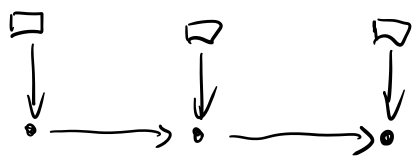
· Slodowy slices ⊂ Springer resolution

Setup: - quiver = (directed) graph $Q=(I,E)$ (assume without loops (most of the time))

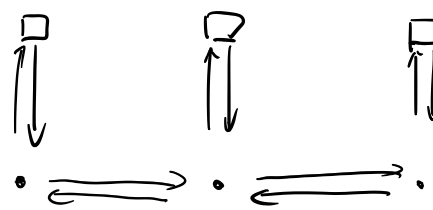
eg 1: 

eg 2: 

· framing



· Take "cotangent space", i.e., double the arrow



Rmk: A few ways of thinking about framing:

1) Nakajima was a differential geometer at one point, studied Gauge theory $\rightsquigarrow$ ADHM equation : $[x,y]+ij=0$
this $+ij$ term only appears when you have framing.

2) Thinking quiver varieties as moduli spaces, framing is like

"marked points" or "bundles with a choice of trivialisation".

3) (practical reason), if no framing, the variety is 0 most of the time.

Nakajima quiver variety.

for every vertice $i \in I$, & framing '$\in Q$', chose a number $\mathbb{N}_{\geq 0}$, i.e. $\underline{v}, \underline{w} \in \mathbb{N}^I$. (Think, $\underline{v}, \underline{w}$ as Hilbert polys?)

The space of all reps of the quiver is:

$$\mathrm{Rep}(\overline{Q^{\heartsuit}}, \underline{V}, \underline{W}) := \bigoplus_{\substack{i \to j \\ j \to i}} \mathrm{Hom}(V_i, V_j) \oplus \bigoplus_{(i) \to \boxed{i}} \mathrm{Hom}(V_i, W_i) \oplus \bigoplus_{(i) \to \boxed{i}} \mathrm{Hom}(W_i, V_i)$$

where $\dim V_i = v_i$

$\dim W_i = w_i$.

There is a $GL(V) = \bigoplus_{i \in I} GL(V_i)$ action on it.

$$g \cdot (x, y, i, j) = (gxg^{-1}, gyg^{-1}, ig^{-1}, gj)$$

There is G-equivariant moment map

$$\mu : \mathrm{Rep}(\overline{Q^{\heartsuit}}, \underline{v}, \underline{w}) \to \mathfrak{g}_V^* \cong \mathfrak{g}_V$$

$$(x, y, i, j) \longmapsto \sum [x, y] + ji \quad (\text{ADHM})$$

So given $\lambda \in Z(\mathfrak{g}_v)$, $\theta : GL(V) \to \mathbb{C}^\times$

Def: $\mathcal{M}_{\lambda,\theta}(Q, \underline{v}, \underline{w}) := \mu^{-1}(\lambda) /\!\!/_{\theta} GL(V)$

We mostly consider the case $\lambda = 0$.

King's stability:

$(x, y, i, j) \in \mu^{-1}(\lambda)$ is θ-semistable

iff $\forall\ S_i \subseteq V_i$ which is stable under the maps x & y, we have

$$S_i \subseteq \ker j_i \quad \forall i \in I \Rightarrow \theta \cdot \dim_I S \leq 0$$

$$S_i \supset \operatorname{Image} i_i, \quad \forall i \in I \Rightarrow \theta \cdot \dim_I S \leq \theta \cdot \dim_I V$$

Example:

$$\begin{tikzcd}
n \\
v_1 \arrow[u] & \bullet \arrow[l] & \bullet \arrow[l] & \bullet \arrow[l] & \cdots & \bullet_{v_r}
\end{tikzcd}$$

$(n > v_1 > \cdots > v_r)$

$\theta = \theta^+ = (1, \dots, 1)$

semistable means that x_i & j are injections

$\underset{[Na]}{\rightsquigarrow}$ $\mathcal{M}_{0,\theta^+} = T^* FL(r, \mathbb{C}^n)$

$T^* Fl(r, \mathbb{C}^n) \longrightarrow X$ is surjective when

$r - v_1 \geq v_1 - v_2 \geq v_2 - v_3 \geq \cdots \geq v_{n-1} - v_n \geq$

$\theta = \underline{0} = (0, \cdots, 0)$

Then any pt is θ-semistable.

What is $\mathfrak{M}_{0,0}$? (some kind of nilpotent orbit closure ...)

$\theta = \theta^- = (-1, \cdots, -1)$

Semistable means that y_i & j are surjections

$\rightsquigarrow \mathfrak{M}_{0,\theta^-} = T^* Fl(r, \mathbb{C}^n)$

but now "flags" are $\mathbb{C}^n \twoheadrightarrow \mathbb{C}^{v_1} \twoheadrightarrow \mathbb{C}^{v_2} \cdots$

$$\begin{tikzcd}
\mathfrak{M}_{0,\theta^+} \arrow[dr] & & \mathfrak{M}_{0,\theta^-} \arrow[dl] \\
& \mathfrak{M}_{0,0} &
\end{tikzcd}$$

Where is the symp alg geo?

The claim is that $\mathfrak{M}_{0,\theta} \to \mathfrak{M}_{0,0}$ is an example of a symplectic singularity, & in many cases, a symplectic resolution.

<u>Def</u>: Let X be affine normal Poisson variety.

$\pi: \tilde{X} \to X$ is a symplectic resolution if $\tilde{X}$ is smooth symplectic s.t. $\pi^* \mathcal{O}_X \cong \mathcal{O}_{\tilde{X}}$ as a poisson algebra, and a resolution of singularities.

Quote: 'Symplectic resolutions are the Lie algebras of the 21st century' — Okounkov.

Properties:

1) Semismall: $\dim(\tilde{X} \times_X \tilde{X}) = \dim X$

Therefore dim of irred components $\leq \dim X$

2) X is a union of finitely many symplectic leaves $X = \bigsqcup X_\alpha$, each X_α is locally closed smooth

3) In the case of a conical symplectic resolution (i.e., that there are $\mathbb{C}^\times$ actions on $\tilde{X}$ and X, such that π is equivariant, and contracts X to a point 0 then $\pi^{-1}(0)$ is a homotopy retract of $\tilde{X}$, and $H^\cdot(\tilde{X}, \mathbb{C}) \cong H^\cdot(\pi^{-1}(0), \mathbb{C})$,

4) More generally, π^{-1}(any point) is isotropic (in the sense of symplectic geo)

When is $\mathfrak{M}_{\lambda,\theta}(v,w) \to \mathfrak{M}_{\lambda,0}$ a symplectic resolution?

Answer: (Almost always) when (λ,θ) is v-regular;

$$(\lambda, \theta) \in \mathbb{C}^I \times \mathbb{Z}^I \subseteq \mathbb{C}^I \times \mathbb{R}^I \cong \mathbb{R}^I \times \mathbb{R}^I \times \mathbb{R}^I$$
$$\cong \mathbb{R}^3 \otimes \mathbb{R}^I$$

Let $R' := \{\alpha \in \mathbb{Z}^I \setminus \{0\} \mid C_Q v \cdot v \leq 2 \quad \forall i \in I\}$

This is the set of roots, when Q is Dynkin or affine Dynkin, this coincides with the usual roots.

C_Q is the cartan matrix, $C_Q := 2I - A_Q$, A_Q is the adjacency matrix.

Back to the example, we had $\bullet\!-\!\!-\!\bullet\!-\!\!-\!\bullet \; - \; - \; - \; \bullet$

$$C_Q = \begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & \ddots & \ddots \\ & & \ddots & 2 \end{pmatrix}$$

and $R' = \{\pm(e_i - e_j)\}$

for $\alpha \in \mathbb{R}^I$, write $\alpha^\perp := \{\lambda \in \mathbb{R}^I \mid \lambda \cdot \alpha = 0\}$

(λ, θ) is v-regular if:

$$(\lambda, \theta) \in (\mathbb{R}^3 \otimes \mathbb{R}^I) \setminus \bigcup_{\{\alpha \in R' \mid 0 < \alpha \leq v\}} \mathbb{R}^3 \otimes \alpha^\perp$$

if $(\lambda, \theta) = (0, \theta^+)$, which is $e_1 \otimes 0 \oplus e_2 \otimes 0 \oplus e_3 \otimes \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$
in $\mathbb{R}^3 \otimes \mathbb{R}^I$

$\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \cdot \alpha \neq 0 \quad \Rightarrow \quad (0, \theta^+)$ (and $(0, \theta^-)$) is v-regular for all v.

So $\mathfrak{M}_{0,\theta^+}(v,w) \longrightarrow \mathfrak{M}_{0,0}$
is a symplectic resolution.

(When $\lambda = 0$), the Weyl group $W (= S_n)$ acts on θ's.
& $\mathfrak{M}_{0,\theta_1} \cong \mathfrak{M}_{0,\theta_2}$ if θ_1, θ_2 in the same chamber.

So, when we were in $\bullet$ (type A_1)
there were 2 chambers $\theta^+ = 1$, $\theta^- = -1$

in $\bullet - \bullet - \bullet \; - - - \; \bullet$ type A_n

there are $(1+n)!$ chambers

There is a $\mathbb{C}^*$ action on the cotangent direction:
$t \cdot (x, y, i, j) = (x, ty, i, tj)$
& the map $\mathfrak{M}_{0,\theta} \to \mathfrak{M}_{0,0}$ is $\mathbb{C}^*$-equivariant.
The point is that $\pi^{-1}(\mathfrak{M}_{0,0}^{\mathbb{C}^*})$ is a lagrangian subvariety.

and in the case when Q has no oriented cycles, $\mathcal{M}_{0,0}^{\mathbb{C}^*} = \{0\}$.
So $\pi^{-1}(0)$ is a Lagrangian in the quiver case.

BM homology

There isn't a notion of fundametal class for non-compact manifolds in usual homology theory, but there is for BM homology.

$$M_1 \times M_2 \times M_3$$

$$\downarrow P_{ij}$$

$$M_i \times M_j$$
$$\cup \text{ closed}$$
$$Z_{ij}$$

$$Z_{12} \circ Z_{23} = P_{13} \circ (P_{12}^{-1} Z_{12} \cap P_{23}^{-1} Z_{23})$$

$$*: \; H_i(Z_{12}) \times H_j(Z_{23}) \longrightarrow H_{i+j-\dim M_2}(Z_{12} \circ Z_{23}) \quad (*)$$

$$C_{12} \quad C_{23} \longmapsto P_{13\,*}\left(\left(C_{12} \boxtimes [M_3]\right) \cap \left(C_{23} \boxtimes M_1\right)\right)$$

Now set $M_i = M$, & $Z = M \times_Y M$ for $\pi: M \to Y$ proper.
This forms an algebra $H_\bullet(Z)$

pick $y \in Y$, $M_y = \pi^{-1}(y)$
Set $M_1 = M_2 = M$, $M_3 = pt$

$$Z_{12} = Z, \quad Z_{23} = M_y, \quad Z_{12} \circ Z_{23} = M_y$$

$$\rightsquigarrow \; H_\cdot(Z) \circlearrowright H_\cdot(M_y)$$

Now back to the quiver case.

Let $\mathcal{M}(w) = \bigsqcup_v \mathcal{M}_{0,\theta^+}(v,w)$

$$\mathcal{M}_0(w) = \bigsqcup_v \mathcal{M}_{0,0}(v,w)$$

$$Z(w) = \bigsqcup_{v,v'} \mathcal{M}_{0,\theta^+}(v,w) \times_{\mathcal{M}_{0,0}(v+v',w)} \mathcal{M}_{0,\theta^+}(v',w)$$

(in other words, $Z(w) = \mathcal{M}(w) \times_{\mathcal{M}_0(w)} \mathcal{M}(w)$)

Let $H_w = H_{top}(Z(w))$

Let $\pi^{-1}_{v,w}(0)$ be the Lagrangian
$$\begin{array}{c} \mathcal{M}_{0,\theta^+}(v,w) \\ \downarrow \pi_{v,w} \\ \mathcal{M}_{0,0} \end{array}$$

$$L_w = H_{top}\Big(\bigsqcup_v \pi^{-1}_{v,w}(0)\Big).$$

Using top as there is a shift in $(\star)$, and semismall property makes sure we stay in top deg. And Lagrangian also has the right dim.

(I think)

$$\rightsquigarrow \quad H_w \circlearrowright L_w$$

Theorem [Na]: There is an algebra map

$$\Phi : \tilde{U}(\mathfrak{g}_Q) \longrightarrow H_W,$$

and L_W is a simple integrable $\mathfrak{g}_Q$-module with highest weight $\sum_{i \in I} w_i \cdot \omega_i$ (ω_i fundamental weight)

When Q is type A, this was first discovered by Ginzburg, "Lagrangain construction of the enveloping algebra $U(sl_n)$"

Define $B_k^{(r)}(v, w) = \{ (V', V'') \mid V'' \in \mathrm{Rep}(\bar{Q}, v + re_k, w),$
$V' \subset V''$ subrep
s.t. $\mathrm{Im}(i_k : W_k \to V_k'') \subset V' \}$

$B_k^{(r)}(v, w)$ is a irreducible component in $Z(v, v+re_k, w)$

Define $E_k^{(r)} = \sum_v [B_k^{(r)}(v, w)]$

let $\Delta(v, w)$ be the diagonal in $\mathfrak{M}_{0,\theta}(v, w) \times \mathfrak{M}_{0,\theta^+}(v, w)$

Then $E_k [\Delta(v, w)] = [\Delta(v - e^k, w)] E_k$

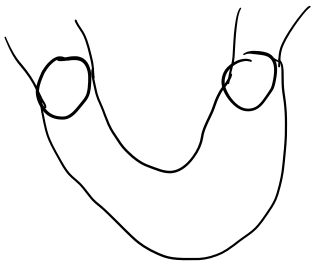
Apparently this is easy to check..

$\mathbb{C}^\times = T^*S^1 \longleftrightarrow \mathbb{C}^\times$

$z = e^{r+i\theta}$

$\omega = dr \wedge d\theta$

$\mathcal{L}_{r\partial_r}\omega = \omega$ Convex



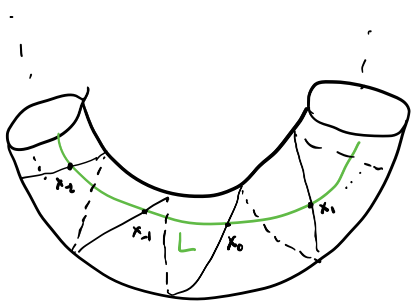
$\mathcal{W}(T^*S^1) \simeq D^b Coh(\mathbb{C}^\times)$

$L \subset T^*S^1$ $\mathcal{O}_{\mathbb{C}^\times}$

$$CW(L,L) \simeq Ext^*(\mathcal{O}_{\mathbb{C}^\times}, \mathcal{O}_{\mathbb{C}^\times})$$

$$\| $$

$$\mathbb{C}[z, z^{-1}] \qquad |z| = 0$$

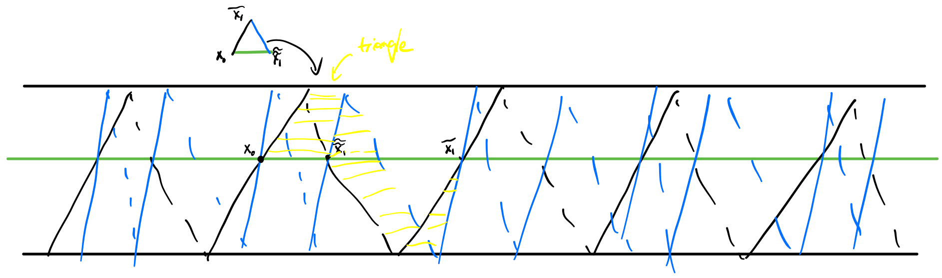


$H: T^*S^1 \longrightarrow \mathbb{R}$

$z \longmapsto \frac{r^2}{2}$

$(z = e^{r+i\theta})$

$$CW(L,L) := \bigoplus_{x \in \phi_H^1(L) \cap L} \mathbb{C} \cdot x$$



blue twice as fast.

x_0

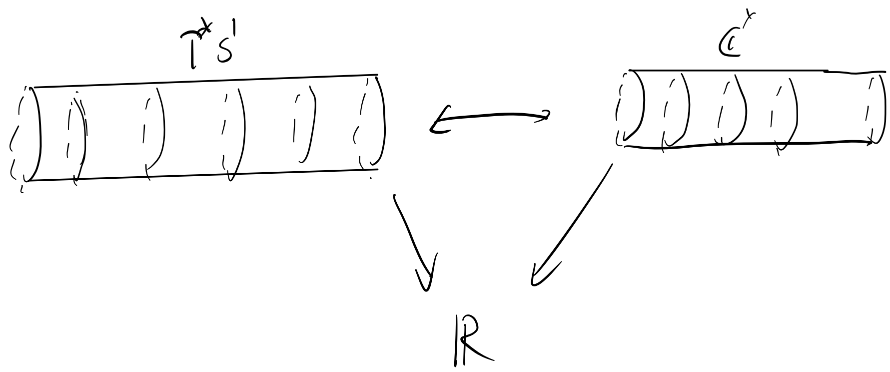
$$CW(L, L) \otimes CW(L, L) \longrightarrow CW(L, L)$$

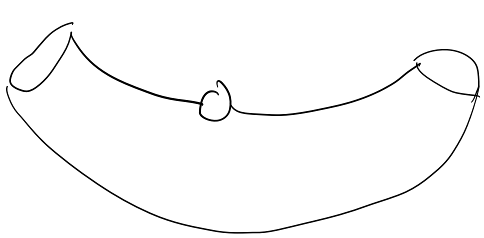
$$\bigoplus \mathbb{C}\cdot\tilde{x} \qquad \bigoplus \mathbb{C}\, x \qquad \bigoplus \mathbb{C}\cdot\tilde{\tilde{x}}$$

$$\tilde{x}\in\phi_H^2(L)\cap\phi_H^1(L) \qquad x\in\phi_H^1(L)\cap L \qquad \tilde{\tilde{x}}\in\phi_H^2(L)\cap L$$

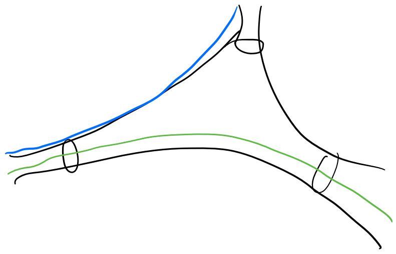
$$\mu^2(x_j, x_i) = x_{i+j} \qquad \mu^2(\tilde{x}_1, x_0) = \tilde{\tilde{x}}_1$$

$$\mathbb{C}[z, z^{-1}] \qquad z^i = x_i$$



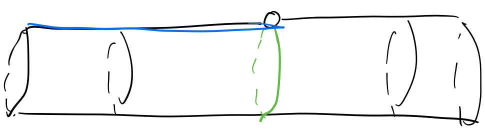


pair of pants

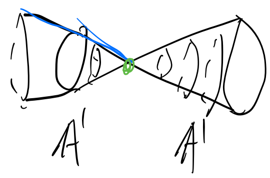


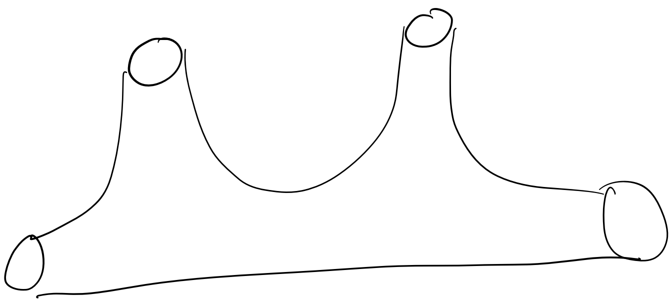
$\longleftrightarrow \quad D^b\mathrm{Coh}\left(\mathbb{C}[z,w]/zw\right)$

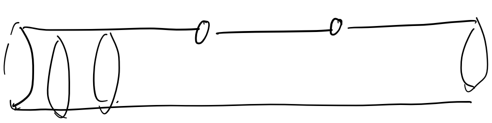
$$\mu^2(x_i, x_j) = \begin{cases} x_{i+j} & \text{if } ij \geq 0 \\ 0 & \text{if } ij < 0 \end{cases} \quad \rightsquigarrow \quad \begin{aligned} z^i &= x_i \\ w^i &= x_{-i} \end{aligned} \quad i > 0$$



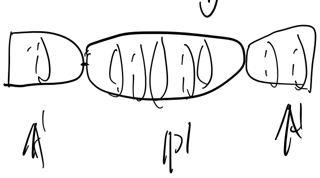
$\longleftrightarrow$

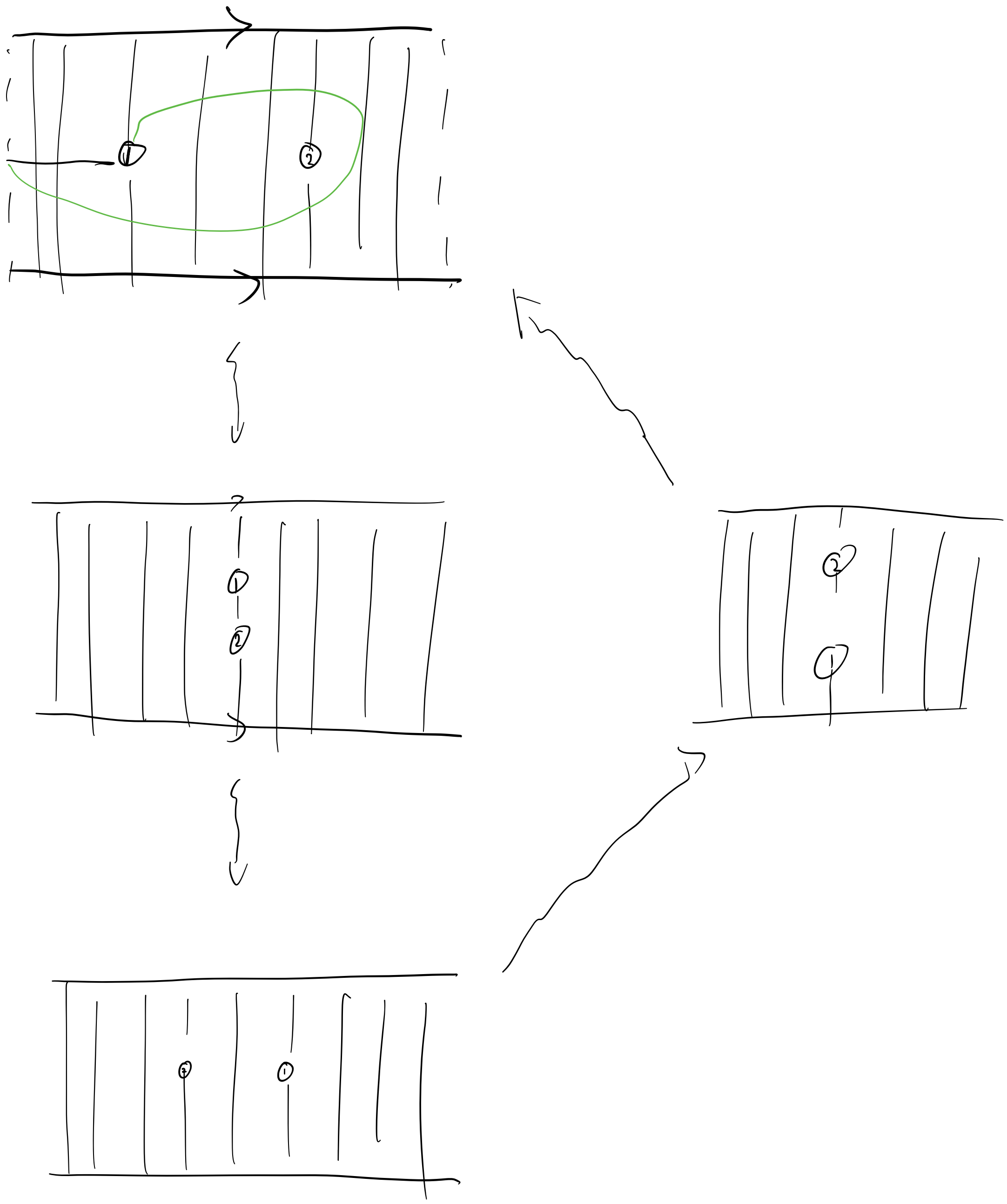


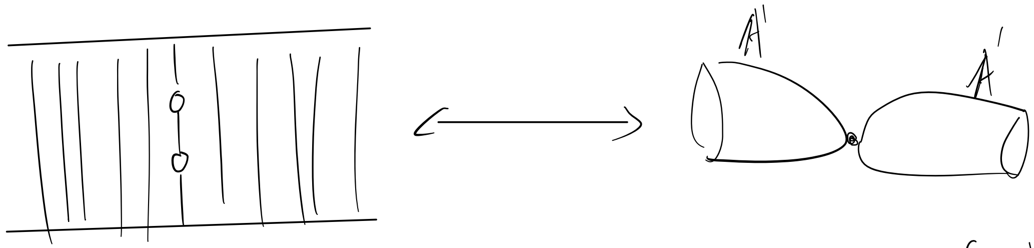


$\mathcal{W}\Big($  $\Big)$

$\Big\downarrow$ SYZ

$D^b\mathrm{Coh}\Big($  $\Big)$





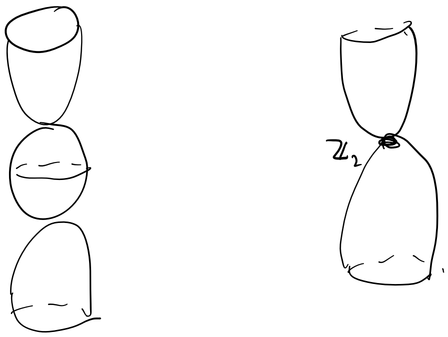
$(x,y) \sim (-x,-y)$

$$\left(\mathbb{C}[x,y]/xy\right) \rtimes \mathbb{Z}_2$$

or $$\left(\mathbb{C}[x,y]/xy\right)/\mathbb{Z}_2$$

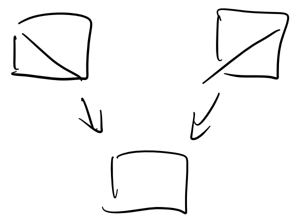
$$\begin{tikzcd}
\widetilde{\mathbb{C}^2/\mathbb{Z}_2} \arrow[d] \arrow[dd, bend right] & \left[\mathbb{C}^2/\mathbb{Z}_2\right] \arrow[dl] \arrow[ddl, bend left] \\
\mathbb{C}^2/\mathbb{Z}_2 \arrow[d, "xy"] & \\
\mathbb{C} &
\end{tikzcd}$$

$$D^b\left(\widetilde{\mathbb{C}^2/\mathbb{Z}_2}\right) \simeq D^b\left(\left[\mathbb{C}^2/\mathbb{Z}_2\right]\right)$$

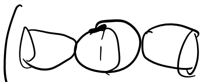


Thm:

$$\begin{tikzcd} Y \arrow[d] & \tilde{Y} \arrow[dl] \\ \mathbb{C}[x,y,w,z]/_{xy-wz} & \end{tikzcd}$$



$$MF(Y, xy) \cong D^b\mathrm{Coh}\left(\quad \right)$$



$\| \; x=y=0$

Siao Chi Mok:

$$\begin{tikzcd} \mathbb{C}^r \arrow[d, shift left, "i"] \\ \mathbb{C}^{n_1} \arrow[u, shift left, "j"] \arrow[r, shift left, "x_1"] & \mathbb{C}^{n_2} \arrow[l, shift left, "y_1"] \arrow[r, shift left, "x_2"] & \cdot \arrow[l, shift left, "y_2"] \cdots \cdot \arrow[r, shift left, "x_{k-1}"] & \mathbb{C}^{n_k} \arrow[l, shift left, "y_{k-1}"] \end{tikzcd}$$

A rep is denoted $(\underline{x}, y, i, j)$.

$\mathrm{Rep}(\bar{Q}^{\heartsuit}, v, w)$

Have moment map $\mu : T^*\mathrm{Rep}(Q^{\heartsuit}, v, w) \longrightarrow \mathfrak{g}_v$ given by

$$(\underline{x}, y, i, j) \longmapsto [\underline{x}, y] + ij$$

$$= \sum_{a \in A} x_a y_a - y_a x_a + ij \quad \in \mathfrak{g}_v = \bigoplus \mathfrak{g}_{v_i} \rightarrow \mathrm{Lie}(GL(v_i))$$

King's stability conditions $\Rightarrow$ (Ginzburg Prop 5.1.5 Cor 5.1.9)

$(\underline{x}, y, i, j) \in \mu^{-1}(0)$ is θ-semistable iff :

For any collection of subspaces $S = (S_i)_{i \in I} \subseteq V = (V_i)$ stable under $\underline{x}, y$, have

$$S_1 \subseteq \ker j \Rightarrow S_i = 0 \ \forall i \in I.$$

We claim that this is equiv to
$j, j \circ x_1 \circ \dots \circ x_i$ injective
$\forall\, 0 \le i \le k$.

Let $\mu(\underline{x}, y, i, j) = 0 \in \mathfrak{g}_V$

so $\sum x_i \circ y_i - \sum y_i \circ x_i + i \circ j = 0 \in \mathfrak{g}_V$

$\Rightarrow \underbrace{-y_1 \circ x_1} + \underbrace{x_1 \circ y_1 - y_2 \circ x_2} + \cdots + x_{k-1} \circ y_{k-1} = \underbrace{-i \circ j}$

(*) $\Rightarrow \underbrace{y_1 \circ x_1 = i \circ j}$, $\underbrace{x_1 \circ y_1 = y_2 \circ x_2}$, $\ldots$, $\underbrace{x_{k-2} \circ y_{k-2} = y_{k-1} \circ x_{k-1}}$, $\underbrace{x_{k-1} \circ y_{k-1} = 0}$.

Claim: If $\ker j \circ y_1 \circ \cdots \circ y_i \neq 0$ for some i, then

$\exists$ non-zero (S_i) stable under x, y, $S_1 \subseteq \ker j$,

$\Rightarrow (\underline{x}, y, i, j)$ not θ^+-semistable.

Pf $\ker j \circ y_1 \circ \cdots \circ y_i \neq 0 \Rightarrow \ker j \neq 0$.

Take $S_1 = \ker j \neq 0$

Note that $y_1 \circ x_1 = i \circ j$

$\Rightarrow \ker j \subseteq \ker y_1 \circ x_1$

$\Leftrightarrow y_1 \circ x_1 (S_1) = 0$.

Want $S_1 \underset{y_1}{\overset{x_1}{\rightleftarrows}} S_2 \underset{y_2}{\overset{x_2}{\rightleftarrows}} S_3 \cdots \rightleftarrows S_k$

Take $S_2 = x_1(S_1)$, $S_3 = x_2(S_2)$ etc.

Then $x_1(S_1) = S_2$, $y_1(S_2) = y_1(x_1(S_1)) = 0 \subseteq S_1$.

Also $x_2(S_2) = S_3$, $y_2(S_3) = y_2(x_2(S_2)) \underset{\circledast}{=} x_1(y_1(S_2)) = 0 \subseteq S_2$

... etc

so (S_i) stable under $\underline{x}, y$, $S_1 = \ker j$

and so $(\underline{x}, y, i, j) \in \mu^{-1}(0)$ not θ^{+} semistable. $\square$

Therefore, $(\underline{x}, y, i, j) \in \mu^{-1}(0)$ θ^{+}-ss iff

j, $j \circ y_1 \circ \cdots \circ y_i$ $\forall i$ are all injective. (in other words $j y_i$ inj $\forall i$)

$$\begin{tikzcd}
\mathbb{C}^r \arrow[d, shift right, "i"] \\
\mathbb{C}^{n_1} \arrow[u, shift right, "j"] \arrow[r, shift left, "x_1"] & \mathbb{C}^{n_2} \arrow[l, shift left, "y_1"] \arrow[r, shift left, "x_2"] & \cdot \arrow[l, shift left, "y_2"] & \cdots \arrow[r, shift left, "x_{k-1}"] & \mathbb{C}^{n_k} \arrow[l, shift left, "y_{k-1}"]
\end{tikzcd}$$

$V_1 = \operatorname{im} j$

$V_2 = \operatorname{im} j \circ y_1$

$\vdots$

$V_k = \operatorname{im} j \circ y_1 \cdots \circ y_{k-1}$

$$\begin{tikzcd}
\mathbb{C}^r \supseteq V_1 \arrow[r, bend left, "x_1"] & \supseteq V_2 & \supseteq \cdots & \supseteq V_k
\end{tikzcd}$$

(Also note that θ^{+}-ss iff θ^{+}-s)

$\circledast \iff i|_{V_1} = x_1, \quad x_1|_{V_2} = x_2$ etc.

$G \curvearrowright (\underline{x}, y, i, j)$ by $(g_j \circ x_j \circ g_{j-1}^{-1},\ g_{j-1} \circ y_j \circ g_j^{-1},\ g_1 \circ i,\ j \circ g_1^{-1})$

$\Big\}$ linear algebra

$\therefore\ \mu^{-1}(0)^{ss}/G = \{ (V_1, \dots, V_k, i, x_1, \dots, x_{k-1}) \mid i|_{V_1} = x_1,\ x_1|_{V_2} = x_2 \text{ etc} \}$

$= \{ (\underbrace{V_1, \dots, V_k}_{\in \mathcal{F} \text{ flag variety.}}, i) \mid \underset{i : \mathbb{C}^r \to \mathbb{C}^r}{i(V_i) \subseteq V_{i+1}} \} =: \tilde{N}.$

by "orbit-stabiliser" (since the action $GL(r, \mathbb{C}) \curvearrowright \mathcal{F}$ is transitive)

Now, $\mathcal{F} = GL(r, \mathbb{C}) / P(\underline{v}) \quad (\underline{v} = (n_1, \dots, n_k))$

Where $P(\underline{v})$ is the stabiliser of the standard flag

$P(\underline{v}) = \{ A \in GL(r, \mathbb{C}) \mid A(V_i) \subseteq V_i \}$,

$V_i = \langle e_1, \dots, e_{n_i} \rangle$ standard flag.

Claim: $T_x(G/P) \cong \mathfrak{g}/Ad_x \cdot \mathfrak{p}$. (Note: G acts on P by right mult.)

Pf: Let $X = G/P$ ($\pi : G \to G/P$) $\mathfrak{p} = Lie(P)$

$T_eG = \mathfrak{g} \to T_xG \to T_x G/P$

$\xi \mapsto R_{x*}(\xi) \mapsto \pi_* R_{x*}(\xi)$

$Ad_x \cdot \mathfrak{p} = (L_x)_*(R_{x^{-1}})_* \cdot \mathfrak{p}$

$\mapsto \pi_* (R_x)_* Ad_x \cdot \mathfrak{p}$

$= (\pi \circ R_x \circ L_x \circ R_{x^{-1}})_* (\mathfrak{p})$

$= D_e(\pi \circ R_x \circ L_x \circ R_{x^{-1}})(D_e(\overbrace{\exp t\mathfrak{p}}^{\in P}))$ $\quad x \cdot \mathfrak{p}$

$= D_e(t \mapsto \underbrace{xP}_{\text{const.}})$

alternatively:

$= 0$ in $T_x(G/P)$ (since P is regarded as a point in G/P)

adjoint action:

$g \cdot \xi = C_{g*}(\xi)$

coadjoint:

$(g \cdot \lambda)(\xi) := \lambda(Ad_{g^{-1}}(\xi))$

So have a map $\mathfrak{g}/Ad_x \cdot \mathfrak{p} \to T_x(G/P)$.

Moreover:

$\pi_* R_{x*}(\xi) = 0 \Rightarrow \pi \circ R_x(\exp t\xi) \equiv \text{const}$ for all small t.

$\Rightarrow (\exp(t\xi)\, x)P = xP$

$\Rightarrow x^{-1}\exp(t\xi)x \in P \Rightarrow Ad_{x^{-1}}(\xi) \in \mathfrak{p}$.

$\Rightarrow \xi \in Ad_x \cdot \mathfrak{p}$.

$\therefore T_x X \cong \mathfrak{g} / Ad_x \cdot \mathfrak{p}$

Claim: $T^*X = \{(x,\lambda) \mid x \in X, \lambda \in Ad_x^* \cdot \mathfrak{p}^\perp\}$

Pf: $T_x^* X = \mathrm{Hom}(\mathfrak{g}/Ad_x \cdot \mathfrak{p}, \mathbb{C})$

$\lambda : \mathfrak{g}/Ad_x \cdot \mathfrak{p} \longrightarrow \mathbb{C}$

iff ~~must satisfy~~ $\lambda(Ad_x \cdot \mathfrak{p}) = 0$ (viewed as $\lambda : \mathfrak{g} \to \mathbb{C}$)

$\Leftrightarrow (\underbrace{Ad_{x^{-1}}^* \cdot \lambda}_{\in \mathfrak{p}^\perp})(\mathfrak{p}) = 0$ $\quad (\mathfrak{p}^\perp := \{\lambda \in \mathfrak{g}^* \mid \lambda|_\mathfrak{p} = 0\})$

$\Leftrightarrow \lambda \in Ad_x^* \cdot \mathfrak{p}^\perp$

$\therefore T_x^* X = \{\lambda \in Ad_x^* \cdot \mathfrak{p}^\perp\}$

$\Rightarrow T^*X = \{(x,\lambda) \mid x \in X, \lambda \in Ad_x^* \cdot \mathfrak{p}^\perp\}$

Recall: $\tilde{N} := \{ (\underbrace{V_1, \dots, V_k}_{\in \mathcal{F}}, i) \mid i(V_i) \subseteq V_{i+1} \} = \mu^{-1}(0)^{ss}/G$.

Claim: (cf. Kirillov p. 181)

We have an isomorphism $\tilde{N} \cong T^*\mathcal{F}$.

Pf: $F = (V_i)$ flag $= g \cdot E_i$, $\mathcal{F} = GL(n, \mathbb{C})/P(\underline{v})$

$\Rightarrow T_g^*\mathcal{F} = \{ \lambda \in \mathfrak{g}^* \mid Ad^*_{g^{-1}} \cdot \lambda(a) = 0 \quad \forall a \in \mathfrak{p} \}$ $\mathfrak{p} \subseteq \mathfrak{g}$, $a \in \mathfrak{g}$

$\lambda(Ad_g(a)) = 0$

identify $\mathfrak{g} \cong \mathfrak{g}^*$ by $\lambda(a) = tr(\lambda^* a)$:

$= \{ b \in \mathfrak{g} \mid tr(b\, gag^{-1}) = 0 \quad \forall a \text{ s.t. } a E_i \subseteq E_i \}$ standard flag.

$= \{ b \in \mathfrak{g} \mid tr(b\, a) = 0 \quad \forall a \text{ s.t. } a V_i \subseteq V_i \}$

We claim that this condition is equiv to $b V_i \subseteq V_{i+1}$.

$(V_{i+1} \subsetneq V_i)$

($\Leftarrow$): If $bV_i \subseteq V_{i+1}$ then $\forall a$ s.t. $aV_i \subseteq V_i$,

if we choose a "compatible basis" for $F = (V_i)$,

then $b = \begin{pmatrix} \boxed{0} & \boxed{\star} & & \star \\ \boxed{0} & \boxed{0} & & \\ & & \boxed{0} & \\ 0 & & & \ddots \end{pmatrix}$

[by compatible, choose basis $v_1, \dots, v_n$ s.t. $v_1, \dots, v_{n_k}$ forms basis of V_k, $v_1, \dots, v_{n_{k-1}}$ ~~~ V_{k-1} etc]

and $a = \begin{pmatrix} \boxed{\star} & & \star \\ & \boxed{\star} & \\ 0 & & \ddots \end{pmatrix}$

$ba = \begin{pmatrix} \boxed{0} & & & \\ & \boxed{0} & & \\ & & \ddots & \\ & & & \boxed{0} \end{pmatrix}$ so $\mathrm{tr}(ba) = 0$.

$(\Rightarrow)$: If $\forall a$ s.t. $aV_i \subseteq V_i$, $\mathrm{tr}(ba)=0$, then:

$$a = \begin{pmatrix} \boxed{\star} & & & \\ & \boxed{\star} & & \\ & & \ddots & \\ & 0 & & \boxed{\star} \end{pmatrix} \quad \text{Suppose } a = \begin{pmatrix} \boxed{\star} & & & 0 \\ & \boxed{\star} & & \\ & & \ddots & \\ 0 & & & \boxed{\star} \end{pmatrix}.$$

$$ba = \begin{pmatrix} \boxed{A_1} & & & \\ & \boxed{A_2} & & \\ & & \ddots & \\ & & & \boxed{A_n} \end{pmatrix} \begin{pmatrix} \boxed{\star} & & & 0 \\ & \boxed{\star} & & \\ & & \ddots & \\ 0 & & & \boxed{\star} \end{pmatrix}$$

$$= \begin{pmatrix} \boxed{A_1\star} & & & \\ & \boxed{A_2\star} & & \\ & & \ddots & \\ & & & \boxed{A_n\star} \end{pmatrix}$$

Note that we can take a to be s.t. one of the $\boxed{\star}$'s is an elementary matrix and the rest of $\boxed{\star}$ zero.

Then $tr(ba) = 0$ for all such a implies that

$A_1 = \cdots = A_n = 0$, and $b V_i \subseteq V_{i+1}$.

$g \cdot (E_i) = (V_i)$.

$\therefore \; T_g^* \mathcal{F} = \{ b \in \mathfrak{g} \mid tr(ba) = 0 \ \forall a \text{ s.t. } a V_i \subseteq V_i \}$

$= \{ b \in \mathfrak{g} \mid b V_i \subseteq V_{i+1} \}$

$\Rightarrow T^* \mathcal{F} = \{ ((V_i), i) \mid i V_i \subseteq V_{i+1} \}$

$= \tilde{N}$.

$\square$