Category & Categorification

plan:
- Categories & Functors
- Adjoints
- Group actions on Categories
- Categorifications
- Examples.

In Statistics / applied maths, quantities are concerned (i.e. find the probability or prove an equation holds)
In pure mathematics, Structures are more often concerned. (i.e. finding two structures are related or the same).
And category theory is the formal way of carrying this out.

In short, pure mathematicians often works (at least) 1 categorical level higher than applied mathematicians.

What is a category:
<u>Def^n</u>: A category $\mathcal{C}$ consists of:
- objects: $o \in \mathcal{C}$
- morphisms/arrows: given C_1, C_2 objects in $\mathcal{C}$,
  $f \in \mathrm{Hom}_{\mathcal{C}}(C_1, C_2)$

Such that morphisms can compose: $f \in \mathrm{Hom}(C_1, C_2)$, $g \in \mathrm{Hom}(C_2, C_3)$
$\exists\ g \circ f \in \mathrm{Hom}(C_1, C_3)$

satisfies: associativity: $(f_1 \circ f_2) \circ f_3 = f_1 \circ (f_2 \circ f_3)$

identity: $\exists!\ e_x \in \mathrm{Hom}(x, x)$

s.t. $e_x \circ f = f \quad \forall f \in \mathrm{Hom}(y, x)$

$g \circ e_x = g \quad \forall g \in \mathrm{Hom}(x, y)$

(= sign here means 'they are the same arrow')

<u>Example:</u> · Grp = category of groups.

objects are groups (i.e. a object is a group)

morphisms are group morphisms

· $\mathrm{Vect}_{\mathbb{K}}$

· Mod_R

· Ring = Category of rings.

· Field = Category of fields.

(This category Field is a bit strange, as the only possible field maps are injections. i.e. field extensions)

<u>More on Categories:</u>

· $\mathbb{K}$-category: Hom-sets are $\mathbb{K}$-vector spaces.

Example: $\mathrm{Vect}_{\mathbb{K}}$

Non-Example: Grp

- (pre-)Additive-category: Given $A, B \in \mathcal{C}$, possible to define $A \oplus B$
  also $f, g \in \mathrm{Hom}(A, B)$, possible to define $f + g$ (via uni property)

  Example: Mod_R

  Non-Example: Field, Ring

- (pre-)Abelian-category: Additive, & given $A \xrightarrow{f} B$, possible to define ker & coker

  Example: Mod_R.

  Non-Example: Ring, Field (has to send 1 to 1)

  Example: Shv_X

  Non-Example: $\mathrm{VectBun}_X$ (finite rank)

  $0 \to \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathcal{O}_{\mathbb{P}^1} \to$ sky scrapper $\to 0$

(Sidenote on $\mathrm{VectBun}_X$, by Serre-Swan 'theorem', vector bundles correspond to f.g projective modules over $\mathcal{O}(X)$, ker/coker $(P_1 \to P_2)$ doesn't have to be proj again).

- Monoidal category: A category where 'tensor product' is possible. bi-additive $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$

- $\mathcal{D}$-Enriched Category: Each $\mathrm{Hom}(c_1, c_2)$ is an object of $\mathcal{D}$.

  Example: $\mathbb{k}$-Categories are exactly $\mathrm{Vect}_{\mathbb{k}}$-Enriched Categories.

  (pre-) Additive Categories are exactly Ab-Enriched Categories

- 2-Category: ? (Enriched over 'Cat')
  i.e. morphisms are categories

Functors:

Functors are maps between categories

$F : \mathcal{C} \to \mathcal{D}$ is an assignment

$F(x)$ $\forall x \in \mathcal{C}$ and $F(f)$ $\forall f \in Hom(x, y)$

s.t. $F(e) = e$

$F(x \cdot y) = F(x) \circ F(y)$

Example: Forgetful functor, Homology, π_1 (not HH^*)

· Cat = category of all (small) categories

morphisms are functors.

Natural transformations:

are maps between functors: $F, G : \mathcal{C} \to \mathcal{D}$

$\eta : F \to G$ is an assignment

$\eta_x : F(X) \to G(X)$ s.t.

$$\begin{tikzcd}
F(X) \arrow[r, "\eta_x"] \arrow[d, "F(f)"'] \arrow[dr, phantom, "\square"] & G(X) \arrow[d, "G(Y)"] \\
F(Y) \arrow[r, "\eta_Y"] & G(Y)
\end{tikzcd}$$

(Think about a map between grp representations)

· $Func(\mathcal{C}, \mathcal{D})$ is a category:

each object is a functor

morphisms are natural transformations,

Thus we see Cat is a 2-category: each hom set

is also a category. The morphisms of the hom sets are called 2-morphisms.
It is possible to define maps between natural transformations and 3-categories...

Another important example:

Mor (Morita) : objects are Mod_A, A a ring

(1-) morphisms are given by bimodules ${}_AM_B$

$$Mod_A \to Mod_B : - \otimes_A M_B$$

2-morphisms are isomorphisms (or homomorphisms) of bimodules

We will think about Mor a lot in this reading seminar (probably)

Adjoints: (F, G) between $\mathcal{C}$ & $\mathcal{D}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$$

are adjoints if $Hom_{\mathcal{D}}(FM, N) \cong Hom_{\mathcal{C}}(M, GN)$
is a isomorphism functorial in both arguments.

- Example: Tensor-Hom adjunction
(All adjunctions I can think of are more or less of this form)

$$Hom(Y \otimes X, Z) \cong Hom(Y, Hom(X, Z))$$

Byproduct is Ind-Res adjunction.

Relation to homological algebra:

right adjoints are always left exact

left adjoints are always right exact.

Therefore $\otimes$ is right exact

& Hom is left exact.

F is left exact if F perserve limits
In particular perserve kernals. I.e.,
if $0 \to M_1 \to M_2 \to M_3 \to 0$ is exact
then $0 \to F(M_1) \to F(M_2) \to F(M_3)$ is exact

G is right exact if G perserve colimits

Exact = left + right exact

Group actions on Categories:

- Groupoid : a category s.t. $\forall f \in \mathrm{Hom}(x, y)$, $\exists ! f^{-1} \in \mathrm{Hom}(y, x)$

  s.t. $f \circ f^{-1} = e_y$
  $f^{-1} \circ f = e_x$

  (i.e. all morphisms are invertible)

  Then $\mathrm{Hom}(x, x)$ is naturally a group

  A groupoid is 'a group with multiple objects'.

Example of groupoid: $\Pi_1(X)$.

objects : points

mors : paths / homotopy

$\pi_1(X, x) = \mathrm{Hom}_{\Pi_1(X)}(x, x)$.

Rmk: One could instead form a 2-Cat where 2-morphisms are homotopies.

Group actions on categories.

Naively: $G \curvearrowright \mathcal{C}$ is

$F(g): \mathcal{C} \to \mathcal{C}$ a functor $\forall g \in G$

s.t. $F(gh) = F(g) \circ F(h)$.

But this is too strict: typically we won't get equality

General principle: $= \rightsquigarrow \cong$

functors themselves live in a category, so we instead ask for: $\eta_{g,h}: F(g)F(h) \cong F(gh)$

a natural isomorphism.

and

$$\begin{tikzcd} F(g)F(h)F(k) \arrow[r, "\mathrm{id}_{F(g)}\eta_{h,k}"] \arrow[d, "\eta_{g,h}\,\mathrm{id}_{F(k)}"'] & F(g)F(hk) \arrow[d, "\eta_{g,hk}"] \\ F(gh)F(k) \arrow[r, "\eta_{gh,k}"] & F(ghk) \end{tikzcd}$$

are equal.

The naive action is the same as $G \longrightarrow \mathrm{Aut}(\mathcal{C})$, map of groups where $\mathrm{Aut}(\mathcal{C})$ is the group of automorphisms of $\mathcal{C}$.

The actual action is the same as $G \longrightarrow \mathrm{Aut}(\mathcal{C})$, a monoidal functor, where G is viewed as a monoidal category with objects elements of G, $\otimes$ = group law, arrows are identities, $\mathrm{Aut}(\mathcal{C})$ viewed as a monoidal cat with $\otimes$ = composition, arrows are natural isomorphisms.

One can also view G as a 2-Cat, with a signal object, arrows = group elements arrow composition = Group law, 2-morphisms = identity. View $\mathrm{Aut}(\mathcal{C})$ as a 2-cat as well, then the action is the same as a 2-functor from G to $\mathrm{Aut}(\mathcal{C})$.

The procedure of going from monoidal categories to 2-categories is called delooping. One can see 2-categories as 'monoidal categories with multiple objects'.

Categorification:

Categorification is the process of replacing set-theoritic theorems with category-theoretic analogues.

replaces	sets	with	categories
	functions		functors
	equations		natural isomorphisms.

The opposite direction is called decategorification.

These are not precise procedures, and there can be many ways of (de) categrifing.

Very often, categorification provides more structures and further insights into the problem.

Infant example: $\underset{\text{Monoidal}}{\mathrm{finVect}_{\Bbbk}}$ categorifies $\underset{\text{monoid}}{\mathbb{N}}$.

$$\begin{array}{ccc} \Bbbk^n & \Bbbk^n \oplus \Bbbk^m & \Bbbk^n \otimes \Bbbk^m \\ \uparrow & \uparrow & \uparrow \\ n & n+m & n \times m \end{array}$$

Decategorification is by taking dim/iso classes/Grothendieck groups.

One sees that $n+m = m+n \rightsquigarrow \mathbb{k}^n \oplus \mathbb{k}^m \cong \mathbb{k}^m \oplus \mathbb{k}^n$

$nm = mn \rightsquigarrow \mathbb{k}^n \otimes \mathbb{k}^m \cong \mathbb{k}^m \otimes \mathbb{k}^n$

In a similar way, $\mathrm{GradedVect}_{\mathbb{k}}$ categorifies polynomials

Knot Theory:

$$\begin{array}{ccc} \text{Khovanov homology} & & \text{Heegaard Floer knot homology} \\ \Big\downarrow \text{categorifies} & & \Big\downarrow \text{Categorifies} \\ \text{Jone polynomials} & & \text{Alexander polynomials} \end{array}$$

A knot is $i: S^1 \hookrightarrow \mathbb{R}^3 / S^3$

A link is a projection of image of i to $\mathbb{R}^2$

i.e. $S^1 \longrightarrow \mathbb{R}^3 \xrightarrow{?} \mathbb{R}^2$

A knot can give many link diagrams, they are related by Skein relations (Reidemeister moves)

Jone polynomials & Alexander polynomials are classical knot invariants (i.e. that $k_1 \cong k_2 \Rightarrow P(k_1) = P(k_2)$)

Khovanov homology of k is a graded vector space s.t. its Euler characteristic $\left(\sum_i (-1)^i \dim V_i\right)$ is Jone poly

Khovanov homology detects the unknot, it is not know if Jone poly does.

In Wei's talk, categorification of reps of Sl_2 helped to construct a derived equivalence of:

$$D^b(T^*Gr(n,k)) \longrightarrow D^b(T^*Gr(n,n-k))$$

Given an (Artinian) abelian category $\mathcal{C}$, we can define it Grothendieck group $K_0(\mathcal{C})$:

It is an abelian group generated by $[C]$, where $C \in ob(\mathcal{C})$, with relations is $0 \to C_1 \to C_2 \to C_3 \to 0$ is a SES, then $[C_2] = [C_1] + [C_2]$

Example: $\mathcal{C}$ = Abelian groups, then $[\mathbb{Z}/p\mathbb{Z}] = 0 \in K_0(\mathcal{C})$
as $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0$

One can similarly define $K_0(F)$, F a functor.

K_0 is a popular way of decategorification.
In Wei's talk, $K_0(F)$ was used to construct the equivalence.

Very similarly, Chuang & Rouquier used a very similar categorification method to solve the abelian defect group conjecture for symmetric groups

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A bit more on K_0

As we said,

$$\begin{array}{c} A\text{-mod} \\ \big\downarrow \text{categorifies} \\ K_0(A) \end{array}$$

However, sometimes we don't gain extra info.
For example, if $A = \mathbb{C}G$, then $\mathbb{C}G\text{-mod} \simeq \mathbb{C}H\text{-mod}$ iff $K_0(\mathbb{C}G) \simeq K_0(\mathbb{C}H)$ (This is a restatement of Main Theorem of Character Table)

But is it possible to recover G or $\mathbb{C}G$ from $\mathbb{C}G\text{-mod}/G\text{-rep}$ or $K_0(\mathbb{C}G)$.

The answer is Yes, and this is the context of Tanaka Duality (one form of it)

Let $F: \mathbb{C}G\text{-mod} \to \text{Vect}$ be the forgetful functor, then $\text{Aut}(F) \simeq G$

One can also view 'Riemann-Hilbert correspondence' $\pi_1(X)\text{-rep} \leftrightarrow Loc(X)$ as an instance of Tanaka duality.