D - module

- D stands for differential operators.
- Motivation from PDEs.
- D-modules form a bridge

Representation theory $\underset{BB}{\overset{\text{Part II}}{\text{———}}}$ D-mod $\underset{RH}{\overset{\text{Part I}}{\text{———}}}$ Algebraic geometry
(of Lie gps / Lie algs) (local systems)

- Part 0 : Intro & motivation
- Part I : Riemann – Hilbert Correspondence (geometric/topological/Rep theoretic interp of D-mods)
- Part II : Beilinson – Bernstein Localisation (Connections to Rep theory of SS Lie alg/gp
- Part III : Kazhdan – Lusztig conjecture (Using geometry to do rep theory)
- Part IV : D-mods on singular spaces.

Notes only

Part 0 :

Field: we work over $\underline{\mathbb{C}}$. X, algebraic variety, smooth

First consider over $\mathbb{C}^n$ (or $\mathbb{A}^n$)

Consider linear partial differential operators :

$$\sum_{i_1, i_2, \dots} f_{i_1,\dots,i_n} \left(\frac{\partial}{\partial x_1}\right)^{\cdot} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\cdots}$$

$f_{i_1,\dots,i_n} \in \mathbb{C}[x_1,\dots,x_n] = \mathcal{O}$

\mathcal{O}

Such element form an algebra (not commutative), $\underline{D}$

multiplication is characterised as

- $\frac{\partial}{\partial x_i} x_i = x_i \frac{\partial}{\partial x_i} + 1$ i.e. $[\frac{\partial}{\partial x_i}, x_i] = 1$ $\quad \left(\frac{\partial}{\partial x}(xg) = g + x\frac{\partial}{\partial x}(g)\right)$ leibniz rule
- $[x_i, x_j] = 0$
- $[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}] = 0$
- $[\frac{\partial}{\partial x_i}, x_j] = 0.$

This is known as $\underline{\text{weyl algebra } W_n}$ on n-variable.

A D-module is a module/rep of W_n.

__Example:__ $D_{\mathbb{A}^1} = W_1 = \mathbb{C}\langle x, \partial \rangle / _{\partial x - x\partial = 1}$.

For general X, We get a $\underline{\text{sheaf}}$, consisting of local data & glueing.

Given a PDE P, naturally we want to find the solutions to P. We can do this using D-modules.

We can associate P a D-module

$\mathcal{M}_P := \mathcal{D}/\mathcal{D}P$ (left module) (not every $\mathcal{D}$-mod is of this form)

Then $\mathrm{Hom}_{\mathcal{D}}(\mathcal{M}_P, \mathcal{O}) = \mathrm{Hom}_{\mathcal{D}}(\mathcal{D}/\mathcal{D}P, \mathcal{O})$

$$= \{\varphi \in \mathrm{Hom}_{\mathcal{D}}(\mathcal{D}, \mathcal{O}) \mid \varphi(P) = 0\}$$

$(\mathrm{Hom}_A(A,B) \simeq B)$ $\simeq \{f \in \mathcal{O} \mid Pf = 0\}$

$\mathrm{Hom}_{\mathcal{D}}(-, \mathcal{O})$ is the Sol_{alg} functor, but the DR functor is used more (more on this later) (solution is in general a local system)

The above construction can be extended to a system of PDEs.

Side note on categories & functors:

Category: • a 'set' of things (objects)
& • maps between them (morphisms)
e.g. $\mathbb{C}[G]$-mod $= G$-reps
Aff_k "affine alg varieties" "=" $\mathrm{Comm}\ k\text{-alg}^{op}$

functor: natural functions between categories:
send objects to objects, morphisms to morphisms
(compatible) (compatible)

e.g. Res, Ind are functors, also pullback of vector bundles.

(takes commuting diagrams to commuting diagrams)

Derived Category: suppose we have

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

then $0 \to Sol(M_3) \to Sol(M_2) \to Sol(M_1) \to Ext^1(M_3, \mathcal{O}) \to \cdots$

Not surjective.

Fix: use derived category and derived functors, the can extend on the right.

objects: consist of complexes of objects

morphism: maps of complexes with quasi-isomorphisms inverted.

Don't worry too much about this, I will only use derived category to state theorems.

$\boxed{\text{End of Part 0}}$

Part I: Riemann-Hilbert

We also want the solution to be finite dimensional.

this leads to a subcategory called holonomic D-modules
(I won't define them)

To study a space X, it is natural to study the category of vector bundle / local system / coherent sheaves on X.

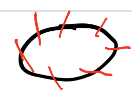
This is like saying to study R, it is natural to

Study R-mod.

A (finite) local system L on X is a locally constant (finite dim) $\mathbb{C}$-linear sheaf: Basically, on every chart we attach a copy of $\mathbb{C}^n$, and they glue nicely. The end result can be non-trivial.

E.g. Thinking of a Mobius band (rank 1 l.s)

$\downarrow$



S^1

$\wr\!\wr$

$\mathbb{C}^* = X$

If X is connected, the local system is equivalent to representations of $\pi_1(X)$ (by monodromy: assign $[\gamma] \in \pi_1(X)$ the operation of moving along γ)

For example: Mobius band $\longleftrightarrow$ $\pi_1(\mathbb{C}^*) \cong \mathbb{Z} \longrightarrow GL_1(\mathbb{C})$

$$n \longmapsto e^{in\pi}.$$

Therefore we have:

$$\{\text{local systems}\} \underset{\text{"deck-transformation"}}{\overset{\text{"take a loop"}}{\rightleftarrows}} \{\pi_1(X)-\text{mod}\}$$

$$\tilde{X} \times \mathbb{C}^m / \pi_1(X)$$

$$\wr\!\wr$$

$$\left\{\begin{array}{c}\text{holonomic D-mod}\\ \text{with regular singularity}\end{array}\right\}$$

Riemann Hilbert upgrade & complete this picture;

(This picture is technically wrong)

More precisely:

$$D\text{-mods} \supseteq \text{hol } D\text{-mods} \supseteq \text{hol, regular singularity} \overset{RH}{\simeq} \text{Perverse sheaves}$$

$$\text{flat connections} \supseteq \text{flat conn, rs} \underset{RH}{\overset{\text{classical}}{\simeq}} \text{Local systems}$$

Example: • $D_{\mathbb{A}^1}/_{x\partial-\lambda} \longleftrightarrow e^{2\pi i\lambda} \rightsquigarrow S^1$ copy of monodromy.

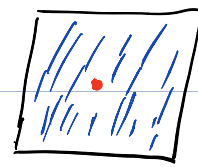
• $\mathbb{C}[\delta]$, not a local system, yet a perverse sheaf, & hol, r.s.

(★) $\left(\mathbb{C}[\delta] = D_{\mathbb{A}^1}/_{D_{\mathbb{A}^1}x} \text{ supported at } 0\right)$

More generally:

$$D^b_{r.h}(D_X-\text{mod}) \xrightarrow[\text{Sol or DR}]{\sim} D^b_c(X)$$

bounded, D-mod, Derived, regular singularity, holonomic, Dual of Sol, constructible.



Important question: What is regular? Let $U = \mathbb{C}\setminus\{0\}$

E.g. $M_1 = D_U/_{D_U(x^2\partial-1)}$, attempt to $Sol(M_1) = \{\lambda e^{\frac{1}{x}} \mid \lambda \in \mathbb{C}\}$

$\not\simeq$

But $M_2 = D_U/_{D_U\partial}$. $Sol(M_2) = \mathbb{C}[x]$, this is not a one to one correspondence.

δ-functions

The Def of regular stop exactly this from happening. M_1 not regular, M_2 is.

Back to RH (★).

Moreover, the equivalence preserve Important functors on both sides. This is known as the six functor formalism.

That is, on both sides there are 6 (derived) functors whenever there is a $f: X \to Y$ (think about $H \leq G$)

They are:

f_* (push forward) (think about Ind)

f^* (pull back) (" " Res)

$f_!$ (exceptional pf)

$f^!$ (- - - pb)

$\mathbb{D}$ (dual)

Hom

I am not going to define them. but some properties: take $p: X \to pt$

$H^i f_* (\mathbb{C}_X) = H^i_{dR}(X, \mathbb{C})$ for X smooth manifold.

$H^i f_! (\mathbb{C}_X) = H^{BM}_{n-i}(X, \mathbb{C}) \cong H^{dR}_{n-i}(X, \mathbb{C})$

More precisely it is saying that the RH ($\bigstar$) will send the D-mod theoric of f_* to the local-system theoric of f_*

i.e. RH is functorial w.r.t these operations.

Therefore "to study the geometry of X is the same as studying D-mod on X"

$\boxed{\text{End of part I}}$

Part II: Beilinson — Bernstein localisation theorem.

0 central character.

Statement: $$D_{G/B}\text{-mod} \underset{-\otimes_{U} D_{G/B} := \mathrm{Loc}}{\overset{\Gamma(G/B,\, -)}{\rightleftarrows}} (U_{\mathfrak{g}})^0\text{-mod}$$

G reductive gp 'SL_n'
B Borel '∇'

Γ: Global section (see example later)

$U\mathfrak{g}$: universal enveloping algebra.

This is complicated, we will only define some terms, and do a good example.

reductive gp : like SL_n.

Borel subgp : like ∇

$U\mathfrak{g}$: like $\mathbb{C}[G]$, holds reps of $\mathfrak{g}$

$$U\mathfrak{g} := T^{\bullet}\mathfrak{g} / (xy - yx - [x,y])$$

$\mathfrak{g}$: lie algebra of G (Tangent space of G at e, reps of $\mathfrak{g}$ are closedly related to reps of G)

$(U\mathfrak{g})^0$-mod : the reps where the centre acts by zero.

<u>Example:</u> $G = SL_2(\mathbb{C})$, B = upper triangular.

$G/B = \mathbb{CP}^1$ (G/B is in general a flag variety)

$\mathfrak{g} = sl_2 = \mathbb{C}\langle e, h, f\rangle$

Traceless matrices

with $[h,e] = 2e$
$[h,f] = -2f$
$[e,f] = h$

$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$
$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
$f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$Z(Usl_2) = \mathbb{C}[c]$ where c is a Casimir
$c = \frac{1}{2}h^2 + ef + fe$

So by $(Usl_2)^0$-mod, we mean an sl_2-representation such that c acts by 0.

<u>Recall:</u> All finite-dim reps of sl_2 are given by highest weight.

i.e. $\exists\, v \in V$, s.t. $e \cdot v = 0$

$h \cdot v = \lambda(h) \cdot v$ for some $\lambda \in h^*$

Call them $L(\lambda)$.

These generalize to infinite-dim reps call Verma-modules $M(\lambda)$

These have basis: $v, fv, f^2 v \ldots\ldots$

with action: $e \cdot v = 0$, $h \cdot v = \lambda v$, $f \cdot (f^n v) = f^{n+1} v$.

Relation between $M(\lambda)$ & $L(\lambda)$.

if $\lambda \in \mathbb{Z}_{\geq 0}$ (alg integral & dominant)

then: $0 \longrightarrow M(-\lambda-2) \longrightarrow M(\lambda) \longrightarrow L(\lambda) \longrightarrow 0$

<u>Remark:</u> There are no finite-dim reps of $\mathcal{D}_{\mathbb{A}^1}$.

How is sl_2/B related to $\mathbb{P}^1$, this is really just

orbit - stabliser: $Sl_2 \longrightarrow \mathbb{P}^1$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto [a : c]$$

$$B \longrightarrow [1:0] = \text{"}\infty\text{"}$$

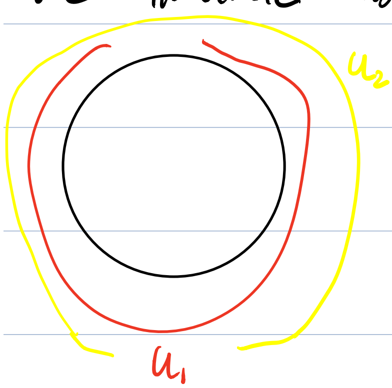
SL_2 acts on $\mathbb{P}^1$ via multiplication

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot [x:y] = \frac{az+b}{cz+d}$$

for $z = \frac{x}{y}$ (Möbius transform)

Since $\mathbb{P}^1$ is not affine, to get algebra & modules, we need to take global section:

We illustrate use example: T_X, $X = G/B$



two charts $U_1 = \operatorname{spec} \mathbb{C}[z]$

$U_2 = \operatorname{spec} \mathbb{C}[w]$

on U_1, $T_{U_1} = \langle z, \frac{d}{dz} \rangle$

U_2, $T_{U_2} = \langle w, \frac{d}{dw} \rangle$

on $U_1 \cap U_2$, we have relation

$$w = z^{-1}$$

$$U_1 \cap U_2 = \operatorname{spec} \mathbb{C}[z, z^{-1}]$$

$$T_{U_1 \cap U_2} = \langle z, z^{-1}, d/dz \rangle$$

$$\begin{tikzcd} T_{\mathbb{P}^1} \arrow[r] \arrow[d] & T_{U_1} \arrow[d] \\ T_{U_2} \arrow[r] & T_{U_1 \cap U_2} \end{tikzcd} \qquad \begin{matrix} z & \frac{d}{dz} \\ \downarrow & \downarrow \\ z & \frac{d}{dz} \end{matrix}$$

$$w \longmapsto z^{-1}$$

$$\frac{d}{dw} \longmapsto \text{“}\frac{d}{d(z^{-1})}\text{''} = \left(\frac{dz^{-1}}{dz}\right)^{-1} \frac{d}{dz} = \left(-z^{-2}\right)^{-1} \frac{d}{dz} = -z^2 \frac{d}{dz}$$

$T_{\mathbb{P}^1}$ is the "biggest" algebra such that the diagram commutes.

Can check $T_{\mathbb{P}^1} = \langle \frac{d}{dz},\ z\frac{d}{dz},\ z^2\frac{d}{dz} \rangle$

If we map $sl_2 \longrightarrow T_{\mathbb{P}^1}$

$$e \longrightarrow -\frac{d}{dz}$$
$$h \longrightarrow -2z\frac{d}{dz}$$
$$f \longrightarrow z^2\frac{d}{dz}$$

This is a Lie algebra map. (In fact isomorphism)

However this doesn't induce

$$U_{sl_2} \longrightarrow \Gamma(X, D_X)$$

because $c = \frac{1}{2}h^2 + ef + fe$ is send to 0.

We instead get:

$$U_{sl_2} / Z(U_{sl_2}) \longrightarrow \Gamma(\mathbb{P}^1, D_{\mathbb{P}^1})$$

Example: δ-module at 0.

$\mathbb{C}[\partial]$, (this is more naturally a right D-module)

action is given by "fourier transform" then multiply:

$$\begin{pmatrix} z \to -\partial \\ \partial \to z \end{pmatrix}$$

$$\cdot \ -\frac{\partial}{\partial z} \cdot \partial^i = -\partial^{i+1}$$

$$\cdot \ -2z\frac{\partial}{\partial z}\partial^i = -2z\partial^{i+1} = +2(i+1)\partial^i$$

$$\cdot \ z^2\frac{\partial}{\partial z}\partial^i = z^2\partial^{i+1} = -z(i+1)\partial^i$$

$$= +i(i+1)\partial^{i-1}$$

Therefore, this form a rep of lowest weight 2.

with lowest weight vector "$1 = \partial^0$"

The dual of this rep is M_{-2}, verma module

of height weight -2.

$\boxed{\text{End of Part II}}$