Research Statement

I am interested in representation theory and its interaction with geometry. In particular, I am interested in algebraic D-modules and their applications. My work fuses techniques from (derived) algebraic geometry, Poisson geometry and symplectic resolutions. It has implications for the general theory of D-modules on singular varieties and for Poisson homology (resp. Hochschild homology) of Poisson algebras (resp. their quantisations). The questions I research impact mathematical physics, in particular, 3D mirror symmetry.

Part I: D-modules

A D-module is a sheaf of modules over the sheaf $\mathrm{Diff}(X)$ of Grothendieck differential operators on a 'space' X (complex manifold, scheme, stack etc). It was first introduced by Mikio Sato as an algebraic way of encoding systems of differential equations, a field now known as algebraic analysis. The foundations of D-module theory was laid down by Masaki Kashiwara in his master thesis and independently by Joseph Bernstein. Methods of tackling D-modules include powerful tools such as homological algebra and sheaf theory. D-modules and their applications are fundamental to representation theory and mathematical physics. In particular, the theory of regular holonomic D-modules and their solution complexes form a key part of the Riemann–Hilbert correspondence which gives a sophisticated answer to Hilbert's 21st problem. This answer has a close relationship to perverse sheaves and intersection cohomology of stratified spaces. Moreover, in representation theory, the Beilinson–Bernstein localization theorem relates D-modules on flag varieties G/B to representations of the Lie algebra $\mathfrak{g}$ attached to a reductive group G. This result opened totally new perspectives, resolving questions such as the Kazhdan–Lusztig conjectures.

While on a smooth variety X, a D-module can be defined just as a module over Grothendieck's ring of differential operators $\mathrm{Diff}(X)$, the situation becomes complicated when X is singular. In particular $\mathrm{Diff}(X)$ can be poorly behaved (*e.g.*, not Noetherian) when X is not smooth. There are alternative approaches available which coincide for smooth varieties but give a nicer category in general. For example, Kashiwara defined D-modules on an (affine) variety X via a closed embedding $X \hookrightarrow Y$ into a smooth variety Y. Grothendieck defined the notion of crystals, an infinitesimal analogue of parallel transport. These two approaches are known to be equivalent and yield a natural category of D-modules on any variety, which enjoys nice properties.

In my work [Yan21], I correct the ring of differential operators, giving a new approach to D-modules on singular spaces. Using a compact generator D_X from the crystal approach, I show that the derived category of D-modules on X is equivalent to the category of DG-modules over a DG algebra $\mathrm{REnd}(D_X)$. The zeroth cohomology, $\mathrm{End}(D_X)$, of this DG algebra is isomorphic to $\mathrm{Diff}(X)$. Hence it can be viewed as a DG correction to $\mathrm{Diff}(X)$. In some cases, this DG algebra really *is* (quasi-isomorphic to) the ordinary algebra $\mathrm{Diff}(X)$. This includes the cuspidal case considered in [BZN04] in the sense that when X is cuspidal, this derived equivalence of D-modules is the derived version of the abelian equivalence in [BZN04]. It is an interesting question if the converse holds, *i.e.*, that $\mathrm{REnd}(D_X) \cong \mathrm{Diff}(X)$ implies X is cuspidal. In any case, for most X, the DG correction to $\mathrm{Diff}(X)$ is nontrivial.

I considered many special cases where it is possible to compute the DG algebra and its action on D-modules explicitly. In the case of a hypersurface $X = \{f = 0\} \subset \mathbb{A}^n$, $\mathrm{Ext}^\bullet(D_X, D_X)$ is concentrated in degree 0 and 1. So D-modules are equivalent to modules over the ordinary algebra $\mathrm{Diff}(X)$ if and only if

$$\frac{D_{\mathbb{A}^n}}{fD_{\mathbb{A}^n} + D_{\mathbb{A}^n}f} = 0.$$

In particular, the vanishing holds for the cuspidal case, which seems difficult to prove directly. When the variety is a curve C, by calculating $\mathrm{Ext}^1(D_C, M)$, I have shown that the abelian subcategory of regular holonomic D-modules with '*completely non-trivial monodromy*' around each non-cuspidal singularity over a curve maps to ordinary modules over $\mathrm{Diff}(C)$, *i.e.*, they have no higher cohomology. I expect the converse to hold for *simple* D-modules and proved this in some cases. Here, *'completely nontrivial monodromy'* means that, in the normalisation of C, all eigenvalues of monodromies about exceptional points are not equal to 1. For an isolated finite quotient singularity $X = V/G$, I have shown that local systems correspond to ordinary $\mathrm{Diff}(X)$-modules if there is trivial monodromy about singularities. The converse holds for simple D-modules, or more generally, intersection cohomology D-modules.

In the future, I would like to explore the A_∞ structure on $\mathrm{Ext}^\bullet(D_X, D_X)$, in particular, **I would like to investigate under what condition the A_∞ structure on $\mathrm{Ext}^\bullet(D_X, D_X)$ is formal.** This condition would allow me to study D-modules as modules over a graded (rather that DG) ring.

Part II: Hochschild–de Rham homology

In [ES09], Etingof and Schedler has developed a new theory of Poisson homology called Poisson–de Rham homology combining ideas from Poisson geometry and D-modules. It is defined as the $(-i)$-th cohomology of the derived D-module-theoretic pushforward to a point of a natural D-module $M(X)$ on X, which captures the Hamiltonian flow induced by the Poisson structure. In other words,

$$HP_i^{DR}(X) := H^{-i}\pi_* M(X),$$

where $\pi : X \to \mathrm{Spec}(\mathbb{C})$. They have shown that $HP_0^{DR}(X) \cong HP_0(X)$. Moreover, they have shown that in nice situations such as when X has finitely many symplectic leaves, $HP_i^{DR}(X)$ is finite dimensional, hence $HP_0(X)$ is finite dimensional. This includes finite quotient singularities (originally proved in [BEG02] by Berest, Etingof and Ginzburg, answered a question of [JF03] of Alev and Farkas).

Going from classical mechanics to quantum mechanics, it is natural to consider deformation quantisations of Poisson algebras, and these quantisations naturally have associated Hochschild homologies. Using ideas from [ES09], I used D-modules to define a new version of Hochschild homology called Hochschild–de Rham homology and shown that $HH_0^{DR}(X) \cong HH_0(X)$. It is defined as

$$HH_i^{DR}(X) := H^{-i}\pi_* M_\hbar(X),$$

where $M_\hbar(X)$ is the a natural "quantised version" of $M(X)$. I have shown that Hochschild–de Rham homology behaves nicely in the situation of symplectic resolutions. In particular, I have shown if X has a symplectic resolution $\rho : \tilde{X} \to X$, and under the assumption that the natural quantisation D-module $M(X)$ is isomorphic to $\rho_*\Omega_{\tilde{X}}$, the D-module-theoretic pushforward of the canonical sheaf on the resolution, then $HH_i^{DR}(A)$ is independent of A, where A is a quantisation of $\mathcal{O}(X)$ and $HH_0(A) = H^{\text{top}}(\tilde{X})$ (generalising Nest-Tsygan theorem).

While the assumption that $M(X) \cong \rho_* \Omega_{\tilde{X}}$ has been shown to be false in a particular case of a quiver variety with loops in [Tsv19], one can consider a "quantised version"

$$M_\hbar(X)[\hbar^{-1}] \cong \rho_* \Omega_{\tilde{X}}(\!(\hbar)\!), \tag{†}$$

where ρ_* now is the D-module-theoretic pushforward for a formal family. And this now has a chance to hold for *all* symplectic resolutions. The goal is then to prove (†) and then deduce the classical case $M(X) \cong \rho_* \Omega_{\tilde{X}}$ for quiver varieties without loops. Here, (Nakajima's) quiver varieties are moduli spaces of representations of quivers (*i.e.*, directed graphs). These are notable for allowing geometrical constructions of the universal enveloping algebra of Kac-Moody algebras acting on the cohomology of quiver varieties and related spaces. Moreover, in [PS17a], it has been proven that, if for every symplectic leaf S of X, a formal slice X_S to leaf at a point $s \in S$ satisfies $HP_0(X_S) \cong H^{\dim X_S}(\rho^{-1}(s))$, then $M(X) \cong \rho_* \Omega_{\tilde{X}}$ is true. **I would like to prove a similar statement in the Hochschild case and use it to prove** $M_\hbar(X)[\hbar^{-1}] \cong \rho_* \Omega_{\tilde{X}}(\!(\hbar)\!)$.

Kontsevich and others have proved that every smooth affine Poisson variety has a (formal) deformation quantisation, called the Kontsevich quantisation. In fact, there is an L_∞ quasi-isomorphism,

$$T_{poly}(X) \xrightarrow{\sim} D_{poly}(X),$$

where $T_{poly} := \bigwedge^\bullet_{\mathcal{O}(X)} \operatorname{Vect}(X)[1]$ is the dgla of (shifted) polyvector fields on X, and $D_{poly} := C^\bullet(\mathcal{O}(X))[1]$ is the dgla of (shifted) Hochschild cochains on X, where the chain maps are restricted to those are differential operators. By taking the the Maurer Cartan twist of this L_∞ quasi-isomorphism with respect to the MC element corresponding to the (formal) Poisson structure, it can be shown that $HH_i(A) \cong HP_i(X)$ where A is the Kontsevich quantisation. **I would like to prove the de Rham version is also true:** $HH_i^{DR}(A) \cong HP_i^{DR}(X)$.

In [PS17b], Pym and Schedler have found a complex of D-modules $\mathfrak{M}_X$ that governs the ordinary Poisson cohomology of X. Moreover, the zeroth cohomology of the complex $\mathfrak{M}_X$ is a quotient of the canonical D-module M_X. This links the ordinary Poisson cohomology with the Poisson-de Rham homology. **I would like to generalise the above construction to the Hochschild-de Rham setting and define Hochschild–de Rham cohomology**. I would like to apply Hochschild-de Rham (co)homology to study deformation theory and ordinary Hochschild (co)homology of symplectic singularities and their quantisations (*e.g.*, quantisations of $\operatorname{Sym} Y$, where Y is a affine symplectic surface, which would be a global analogue of symplectic reflection algebras).

Part III: Quantum Topology

As an application of the Hochschild-de Rham theory, in the upcoming joint work with Sam Gunningham, David Jordan and Monica Vazirani, we derive a closed formula for the dimension of the Skein module of the 3-torus $SK_{SL(n)}(T)$. This uses the fact that the dimension of the skein *algebra* of the 2-torus is equal to the 0-th Hochschild homology of an algebra that is a deformation quantisation of the character variety which admits a symplecic resolution. This symplectic resolution is a version of a Hilbert scheme that turns out to be a finite covering of the usual Hilbert scheme of the

2-torus. Generalising the work of Nakajima and Grojnowski, it becomes a combinatorial task of calculating the top cohomology of this space. And finally by the *generalised Nest-Tsygan theorem*, since it is a symplectic resolution, it does not matter what the quantisation actually is, and HH_0 is always the top de Rham cohomology, and hence we obtain the dimension of

$$\dim SK_{SL(n)}(T) = \mathcal{P} \star J_3(n),$$

where $\mathcal{P}$ is the number of partitions function, J_3 is the 3rd Jordan function and $\star$ is the Dirichlet convolution. This generalise the result of Carrega [Car17] and Gilmer [Gil16] for the $\dim SK_{SL(2)}(T) = 9$ case. **I would like to generalise this approach to $\Sigma_g \times S^1$ and possibly to (-1)-shifted symplectic structure settings.**

Part IV: Coulomb Branch

Let G be a complex reductive group and M a symplectic representation. Often, we will start with any representation V and make it symplectic by setting $M = T^*V$ with the usual symplectic form. Attached to this datum is a 'supersymmetric theory', and its Higgs and Coulomb branches.

The Coulomb branch M_C was defined by Nakajima, Braverman and Finkelberg in [BFN19]. It is defined as the spectrum of a ring constructed as a convolution algebra in the homology of the affine Grassmannian,

$$M_C := \operatorname{Spec}(H_\bullet^{G(\mathcal{O})}(\mathcal{R}_{G,T^*V}), *),$$

where $\mathcal{R}_{G,T^*V}$ is moduli space of triples $(\mathcal{P}, \phi, s)$, where $\mathcal{P}$ is a G-bundle on the formal disk $D = \operatorname{Spec}\mathbb{C}[[z]]$, ϕ is a trivialization of $\mathcal{P}$ over the punctured disk $D^* = \operatorname{Spec}\mathbb{C}((z))$ and s is a section of the associated vector bundle $P \times_G T^*V$ such that it is sent to a regular section of a trivial bundle under ϕ.

The Poisson structure of the Coulomb branch comes from its natural non-commutative deformation $(H_\bullet^{G(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathcal{R}_{G,T^*V}), *)$. In the case of (framed) ADE quivers, it has been shown that M_C is isomorphic to generalised affine grassmannian slices W_μ^λ. And they admit symplectic resolutions if λ is a sum of minuscule coweights, which is automatic if the quiver is type A. Moreover, in [Web19], Webster has shown that $M_H(G, T^*V)$ and $M_C(G, T^*V)$ satisfy a symplectic duality property. In particular, there is a Koszul duality between generalizations of category $\mathcal{O}$ over quantisations of these varieties, if M_H is a Nakajima quiver variety or smooth hypertoric variety.

I would like to study D-modules on Coulomb branches, especially in the ADE quiver case. In particular, I want study the natural D-modules $M(X)$ and $M_\hbar(X)$ and therefore study the Poisson(-de Rham) homology and Hochschild(-de Rham) homology of M_C and its quantisation. I would like to apply the techniques from the previous part in this setting, and tackle these questions I asked in the previous part in this special case. I would like to investigate how symplectic duality exchanges both sides on the level of D-modules.

All my publications are posted on arXiv in preprint form, contributing to the culture of open and free access to science.

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