Seidel — Thomas

Braid group actions on derived categories of coherent sheaves.

Goal: construct Braid group actions on $D^b(X)$ for some varieties X (for example Calabi-Yau).

This action is faithful for $\dim X \geq 2$.

i.e., $B_n \longrightarrow \mathrm{Auteq}(D^b(X))$ injective.

My friend Jian Zhou has helped me a lot with HMS stuff, He is currently a student of Sheel Ganatra at USC, and working toward braid group actions on perverse sheaves on Weinstein manifolds, He mainly works on the A-side.

Motivation: How did they come up with this?

· Homological mirror symmetry. — We will spend quite some time on this.

· Although, most of the proofs don't use HMS and handle $D^b(X)$ directly.

Examples: · Resolutions of ADE singularities

· $\begin{array}{c}\tilde{\mathfrak{g}}\\ \downarrow\\ \mathfrak{g}\end{array} \Big/ \begin{array}{c}\tilde{\mathcal{N}}\\ \downarrow\\ \mathcal{N}\end{array}$. 中島 Nakajima quiver varieties (talk on this later).

· many more examples in the paper.

Addendum: $\mathrm{Auteq}(D^b(X))$ alway contains $A(X) = (\mathrm{Aut}(X) \ltimes \mathrm{Pic}(X)) \times \mathbb{Z}$

when ω_X or $\underbrace{\omega_X^{-1}}_{\text{Fano}}$ is ample, then $\mathrm{Auteq}(D^b(X)) = A(X)$.

What is HMS?

MS originated from string theory, which I don't understand...
It says something like: say more, 4+6 = 10, 6 symp or comp, doesn't affect, not detectible, hence expect equivalence.
There are pairs (X, M) of two spaces, such that, while X, M have very different geometries, but are "equivalent" when employed as "extra dimensions of string theory".

More mathematically, Kontsevich formalized HMS as:
(X, M), X = complex alg variety,
M = real symp manifold,
Both compact Kähler Calabi–Yau (i.e. $\Lambda^{top} T^* M \cong \mathcal{O}_M$)

Such that $D^b(X) \cong D^b \mathrm{Fuk}(M, \beta)$ (*)
an equivalence of triangulated categories.

· Say more: SYZ

Some remarks: · $D^b(X)$ is derived category of coherent sheaves on X.
· May my wording wasn't great there, this is saying $\forall X, \exists (M, \beta)$ & conversely $\forall (M, \beta), \exists X$ s.t. (*) is true.
· though I think not $\exists !$ (as far as I know)
· (*) carries a lot of information, has a ton of applications in enumerative geometry. But for our purpose, we have

$$\mathrm{Auteq}\, D^b(X) \cong \mathrm{Auteq}\, D^b \mathrm{Fuk}(M, \beta) \quad (**)$$

Where Auteq = triangulated equivalences of triangulated categories.
(I don't like the word exact)

- (*) is prove for:
  - When the paper was written, only for

    X = Elliptic curve, M = Torus

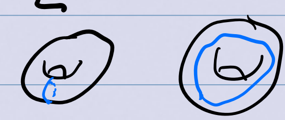
  Now, this has been generalised to:
  - Surfaces — related to gentle algebras
  - Toric manifolds (I think...)
  - fibrations of above (I think...) — using tilting objects

- (*) is believed to hold beyond Calabi-Yau case.

  This is related to Landau-Ginzburg (I think...)

- $D^b Fuk(M, \beta)$ can be attached to any compact symplectic manifold with $c_1(TM) = 0$ ($0 = c_1(\Lambda^{top} T^*M) \Rightarrow 0 = c_1(M)$) Say more
  Seidel-Fukaya cat
  infinitesimal wrapped Fuk cat.
  - partially wrapped Fuk cat

- $D^b Fuk(M, \beta)$, unlike $D(X)^b$, in general is not a derived cat of an abelian cat.
  - it contains objects Lagrangian submanifolds L (with a unitary line bundle U if $\pi_1(M) \neq 0$) (with $\mathbb{Z}$-shifts $L[i]$, $i \in \mathbb{Z}$.)



  - Morphisms defined by: $Hom(L_1, L_2) := HF^0(L_1, L_2) \otimes_{\mathbb{R}} \mathbb{C}$

    This is a graded vector space with the usual:

    $$HF^{*+i}(L_1, L_2) = HF^*(L_1[-i], L_2) = HF^*(L_1, L_2[i])$$

  - Can compose morphisms.
  - Because it is triangulated, it has more objects than $L[i]$, tho it is generated by them.

How does HMS help us?

Recall (**)

$$\mathrm{Auteq}\, D^b(X) \cong \mathrm{Auteq}\, D^b\mathrm{Fuk}(M,\beta).$$

Clearly, if we can handle (subgps of) the RHS, we can find something insightful about the LHS.

Clearly, $\mathrm{Symp}(M,\beta)$, the group of sympletic automorphism acts on M.

Hence $\mathrm{Symp}(M,\beta) \longrightarrow \mathrm{Auteq}\, D^b\mathrm{Fuk}(M,\beta).$

But isotopic automorphisms give the same action, so

$$\pi_0(\mathrm{Symp}(M,\beta)) \longrightarrow \mathrm{Auteq}\, D^b\mathrm{Fuk}(M,\beta).$$

It turns out there is a notion of graded version of symplectic automorphisms in $D^b\mathrm{Fuk}$, denoted $\mathrm{Symp}^{gr}(M,\beta)$, which is a central extension of symp by $\mathbb{Z}$.

Hence

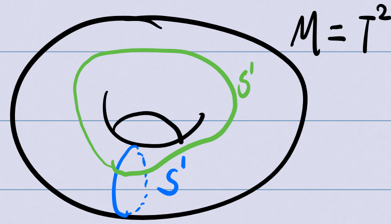
$$\pi_0(\mathrm{Symp}^{gr}(M,\beta)) \longrightarrow \mathrm{Auteq}\, D^b\mathrm{Fuk}(M,\beta).$$

The $\mathbb{Z}$ in $\mathrm{Symp}^{gr}(M,\beta)$ just corresponds to shifts in $D^b(X)$. Hence, composing with (*)

$$\pi_0(\mathrm{Symp}(M,\beta)) \longrightarrow \mathrm{Auteq}(D^b(X))/\text{translations}. \qquad (\dagger)$$

What can we say about the LHS of (†) ?

Let's look at the example :



From previous talks, we know that the $\tau_{S'}$ Dehn twist along S' is a symplectic automorphism, same for $\bigcirc S'$.

$\tau_{S'}$ acts on homology $H_*(M)$:

$$(\tau_{S'})_* x = \begin{cases} x - ([S'] \cdot x)[S'] & \text{if } \dim x = 1 \\ x & \text{otherwise} \end{cases}$$

Here x is a cycle $[S']$ is the fundamental class (i.e. image of $H_1(S') \xrightarrow{i_*} H_1(T^2)$)

$\cdot$ is the intersection pairing

in particular, $(\tau_{S'})_* [S'] = [S'] - [S'] \ldots$

Recall that $\tau_{S'}$ and $\tau_{S'}$ satisfy the braid relation of B_3.

i.e. $\tau_{S'} \tau_{S'} \tau_{S'} \simeq \tau_{S'} \tau_{S'} \tau_{S'}$

Therefore $B_3 \longrightarrow \mathrm{Symp}(T^2)$

Now, we can generalise from T^2 to any symp manifold.

the S^1 and S^1 to so call Lagraingian Spheres S_i

The S_i will have generalised Pehn twises τ_{S_i}

Def: an (A_m)-configuration of Lag spheres $S_1, \ldots, S_m \subset (M, \beta)$, is such that

$$|S_i \cap S_j| = \begin{cases} 1 & |i-j| = 1, \\ 0 & |i-j| \geq 2. \end{cases}$$

Then $\tau_{S_1}, \ldots, \tau_{S_m}$ satisfy the braid relations for B_{m+1}. (This is proved for surfaces at least, I don't know the general result).

Now composed with $(\dagger)$, we have

$$\begin{tikzcd} B_{m+1} \arrow[r, "\text{Dehn twist}"] & \pi_0(\mathrm{Symp}^{gr}(M, \beta)) \arrow[r, "\text{action}"] & \mathrm{Auteq}\, D^b\mathrm{Fuk}(M, \beta) \arrow[d, "\text{HMS}"] \\ & & \mathrm{Auteq}\, D^b(X) \arrow[d, "\text{quotient}"] \\ & & \mathrm{Auteq}\, D^b(X)/\text{translation} \end{tikzcd}$$

Now, the question is, this is a motivation, and we want to prove something. Therefore we need to take a guess of the image, then prove thing directly using alg geo & homological alg.

What should be the image?

Observations: • If S is a Lag sphere, then

$$\mathrm{Hom}^*_{D^b\mathrm{Fuk}}(S,S) = HF^*(S,S) \otimes_{\mathbb{R}} \mathbb{C} \cong H^*(S,\mathbb{C}).$$

If $\mathcal{E}$ corresponds to S in HMS,

then $\mathrm{Hom}^*_{D^b(X)}(\mathcal{E},\mathcal{E}) \cong H^*(S,\mathbb{C})$

cohomology of a sphere, therefore whatever $\mathcal{E}$ is, we should call it spherical object.

• There is known to be a long exact sequence;

$$\begin{tikzcd} HF(\tau_S(L_0), L_1) \arrow[rr] & & HF(L_0, L_1) \arrow[dl] \\ & HF(S, L_1) \otimes HF(L_0, S) \arrow[ul, "{[1]}"] & \end{tikzcd},$$

(proved by Seidel)

which is seemed to be induced by the exact triangle:

$$\begin{tikzcd} \tau_S(L_0) \arrow[dr, "{[1]}"'] & & L_0 \arrow[ll] \\ & HF(S, L_0) \otimes S \arrow[ur] & \end{tikzcd}$$

($\mathrm{Hom}(-, L_1)$ is contravariant)

If τ_S corresponds to $\Phi_{\mathcal{E}}$ in $D^b(X)$, then we should have:

(L_0 corresponds to $\mathcal{F}$)

$$\begin{tikzcd} \Phi_{\mathcal{E}}(\mathcal{F}) \arrow[dr, "{[1]}"'] & & \mathcal{F} \arrow[ll] \\ & \mathrm{Hom}^*(\mathcal{E}, \mathcal{F}) \otimes \mathcal{E} \arrow[ur] & \end{tikzcd}$$

This is the defining property of Fourier — Mukai (向井) transform (FMT) along the element:

$$\begin{tikzcd} & \mathrm{cone}(\mathcal{E}^\vee \boxtimes \mathcal{E} \to \mathcal{O}_\Delta) \in D^b(X \times X) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ D^b(X) & & D^b(X) \end{tikzcd}$$

We can check:

$$\mathrm{FMT}_{\mathrm{cone}}(\mathcal{F}) \overset{\mathrm{def}}{=\!=} (\pi_2)_*(\pi_1^*\mathcal{F} \otimes \mathrm{cone})$$

$$= (\pi_2)_*(\mathrm{Cone}(\pi_1^*\mathcal{F} \otimes (\mathcal{E}^\vee \boxtimes \mathcal{E}) \longrightarrow \pi_1^*\mathcal{F} \otimes \mathcal{O}_\Delta) \quad \begin{pmatrix} \otimes \text{ is derived,} \\ \text{hence resp triangles} \end{pmatrix}$$

$$= (\pi_2)_*(\mathrm{Cone}(\pi_1^*\mathcal{F} \otimes \pi_1^*\mathcal{E}^\vee \otimes \pi_2^*\mathcal{E} \longrightarrow \pi_1^*\mathcal{F} \otimes \mathcal{O}_\Delta) \quad \begin{pmatrix} \text{Converting outer } \boxtimes \\ \text{to inner } \otimes \end{pmatrix}$$

$$= (\pi_2)_*(\mathrm{Cone}(\pi_1^*\mathcal{H}om(\mathcal{E},\mathcal{F}) \otimes \pi_2^*\mathcal{E} \longrightarrow \pi_1^*\mathcal{F} \otimes \mathcal{O}_\Delta) \quad (\text{easy to see}\ldots)$$

$$= \mathrm{Cone}(\mathcal{H}om(\mathcal{E},\mathcal{F}) \otimes \mathcal{E} \longrightarrow \mathcal{F}) \quad \begin{pmatrix} (\pi_2)_* \text{ is triangulated} \\ \text{hence resp triangles} \\ + \text{ projection formula} \end{pmatrix}$$

$$\cong T_{\mathcal{E}}(\mathcal{F}) \quad \begin{pmatrix} \text{Axiom of} \\ \text{triangulated cat} \end{pmatrix}$$

(This is not good enough, because I am using local freeness of $\mathcal{E}$) But this is serving a justification of why Dehn twists corresponds to FMTs under HMS)

We will see examples when $\mathcal{E}$ is locally free.

We now abuse notation, we write $T_{\mathcal{E}}$ to be $\mathrm{FMT}_{\mathrm{cone}(\mathcal{E}^\vee \boxtimes \mathcal{E} \to \mathcal{O}_\Delta)}$.

- It is known that $(T_{\mathcal{E}})_*(y) = y - \langle [\mathcal{E}], y \rangle [\mathcal{E}]$ for $y \in K_0(X)$, $\langle -,- \rangle$ the Euler pairing.

Now compare this with:

T_{S^1} acts on homology $H_*(M)$:

$$(T_{S^1})_* x = \begin{cases} x - ([S^1]\cdot x)[S^1] & \text{if } \dim x = 1 \\ x & \text{otherwise} \end{cases}$$

Here x is a cycle $[S^1]$ is the fundamental class (i.e. image of $H_1(S^1) \xrightarrow{i_*} H_1(T^2)$)

$\cdot$ is the intersection pairing

in particular, $(T_{S^1})_* [S^1] = [S^1] - [S^1]$ ....

This is enough evidence, and motivate the following definition:

<u>Def</u>: $\mathcal{E} \in D^b(X)$ is spherical if (X an alg variety)

- $\mathrm{Hom}^*(\mathcal{E}, \mathcal{E}) \cong H^*(S^{\dim X})$
- $\mathcal{E} \otimes \omega_X \cong \mathcal{E}$ (More on this later, obvious don't need this if X is Calabi-Yau. This is related to Serre duality.)
- an (A_m)-config is $\mathcal{E}_1, \ldots, \mathcal{E}_m$ s.t.

$$\dim_{\mathbb{C}} \mathrm{Hom}^*_{D^b(X)}(\mathcal{E}_i, \mathcal{E}_j) = \begin{cases} 1 & |i-j| = 1 \\ 0 & |i-j| \geq 2 \end{cases}$$

<u>Theorem</u> (ST) $\cdot$ $T_{\mathcal{E}}$ is an exact self equivalence

- If $\mathcal{E}_1, \ldots, \mathcal{E}_m$ is an (A_m)-config, then $T_{\mathcal{E}_i}$ satisfy Braid relations of B_{m+1}
- if $\dim X \geq 2$, this is faithful.

ST proved this without using HMS

Therefore conjecture:

$$\begin{tikzcd} B_{m+1} \arrow[r, "\text{Dehn twist}"] \arrow[dr, bend right, "ST"'] & \operatorname{Auteq} D^b \operatorname{Fuk}(M, \beta) \arrow[d, "HMS"] \\ & \operatorname{Auteq} D^b(X) \end{tikzcd}$$

If HMS is true, then for $\dim X \geq 2$, the Dehn twist action of B_{m+1} should also be faithful.

Now, we are going to forget HMS for some time, & talk about some aspects of the proofs.

There are technical subtleties such as replace $F \in D^b(X)$, by a complex of injectives, which technically doesn't live in $D^b(X)$ anymore, but we are going to ignore issues like this. (can be resolved by doing more homo alg.)

We have defined $\Phi_\mathcal{E}(F) := \operatorname{Cone}(\hom(\mathcal{E}, F) \otimes \mathcal{E} \to F)$.
The claim is $\Phi_\mathcal{E}$ is invertible.
The candidate of inverse is given by $\Phi'_\mathcal{E}(F) := \operatorname{Cone}(F \to \operatorname{Linhom}(\hom(F, \mathcal{E}), \mathcal{E}))$

If $\cdot\ \operatorname{Hom}^i(E, E) = \begin{cases} k & \text{if } i = 0, k \\ 0 & \text{otherwise} \end{cases}$

• $\mathrm{Hom}^j(F, E) \times \mathrm{Hom}^{n-j}(E, F) \longrightarrow \mathrm{Hom}^n(E, E) \cong k$ (*)
is non-degenerate

Then Φ_E, Φ'_E are inverses.

Note here, we are not using the fact we are working in $D^b(X)$, any triangulated category is fine.
The proof uses a lot of tensor-hom adjunction, and homological algebra.

The second (*) is equivalent to $E \otimes \omega_X \cong E$, if we work in $D^b(X)$ and X is smooth projective. This is pretty much a restatement of Serre duality (alg geo version of poincare duality)

if X is quasi-projective, then need to the dualising sheaf $p^! \mathcal{O}_{pt}$ rather than ω_X, and this becomes a restatement of Grothendieck duality.

About braid relations

I think everything follows from the key lemma:
If E_2 is (n-)spherecial, then

(**) $\Phi_{\Phi_{E_2}(E_1)} \Phi_{E_2} \cong \Phi_{E_2} \Phi_{E_1}$, this is proved by unwinding the def. and general abstract non-sense.

From this lemma, all braid relations follow.
For example, if $\mathrm{Hom}^i(E_2, E_1) = 0 \ \forall i$, then $T_{E_1} T_{E_2} \cong T_{E_2} T_{E_1}$

Pf:
$$\mathrm{Hom}(E_2, E_1) \otimes E_2 \longrightarrow E_1 \longrightarrow \Phi_{E_2}(E_1) \xrightarrow{+1}$$
$$\underset{0}{\parallel}$$

$$\Rightarrow \quad E_1 \cong \Phi_{E_2}(E_1)$$

$$\Rightarrow \quad \Phi_{E_1} \cong \Phi_{\Phi_{E_2}(E_1)}$$

$$\Rightarrow \text{ by } (**), \quad \Phi_{E_1} \Phi_{E_2} \cong \Phi_{E_2} \Phi_{E_1}.$$

The other relation can be proved similarly.

Now, the authors of the paper said they didn't bother to check is the action is strong (i.e. in Deligne's sense) or not.

I believe it is, but I didn't bother to check either. This requires a lot of diagram chasing.

Example: • if X is strict Calabi-Yau, i.e., $\omega_X = \mathcal{O}_X$, & $H^i(X, \mathcal{O}_X) = 0$ $0 < i < n$, then any line bundle is a spherical object, just because $Hom(L, L) \cong \mathcal{O}_X$.

• If $Y \subset X$, local complete intersection in smooth proj, with the normal sheaf $\nu = (J_Y / J_Y^2)^\vee$. Assume that $\omega_X|_Y$ is trivial & that $H^i(Y, \Lambda^j \nu) = 0$ $\forall\, 0 < i+j < n$. Then $\mathcal{O}_Y \in D^b(X)$ is a spherical object.

Pf: Koszul resolution $\Rightarrow$ $\mathcal{E}xt^j(i_*\mathcal{O}_Y, i_*\mathcal{O}_Y) \cong i_*(\Lambda^j \nu)$
Spectral sequence gives $Ext^r(i_*\mathcal{O}_Y, i_*\mathcal{O}_Y) = 0$ $0 < r < n$
Duality gives the rest.

• Application of previous point:
Let X be a surface, suppose C is a smooth rational curve C s.t. $C \cdot C = -2$ (self-intersection), then C satisfies the conditions of the previous point.

$C \cdot C = -2$ is saying $PD(C) \wedge PD(C) = -2[X]$
PD is the the Poincare dual of $H_*(C) \longrightarrow H_*(X)$

Fact: $C \cdot C = \deg(\nu)$ the deg of normal sheaf

Explain in words here ... so $\deg(\nu) = -2$
$\Rightarrow H^i(C, \Lambda^j \nu) = 0$

Adjunction formula $\Rightarrow$ $\omega_X|_C = \mathcal{O}_C$.

& if $C_1, \dots, C_m$ of such curves
& $C_i \cap C_j = \phi$ for $|i-j| \geq 2$ $(\Rightarrow \mathrm{Hom}(\mathcal{O}_{C_i}, \mathcal{O}_{C_j}) = 0)$
& $C_i \cap C_{i+1} = 1$ for $i = 1, \dots, m-1$ $(\Rightarrow \mathrm{Hom}(\mathcal{O}_{C_i}, \mathcal{O}_{C_{i+1}}) = \mathbb{C})$
then $(\mathcal{O}_{C_1}, \dots, \mathcal{O}_{C_m})$ is an A_m-config

- What is a concrete example of the above?
  Where do we find these -2 curves?

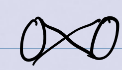
<u>Example:</u> Let $G = C_{m+1} = \mathbb{Z}/_{m+1}\mathbb{Z} \hookrightarrow SL_2(\mathbb{C})$ acting on $\mathbb{A}^2$, then $X = \mathbb{A}^2/G = \mathrm{Spec}(\mathbb{C}[x,y]^G)$.

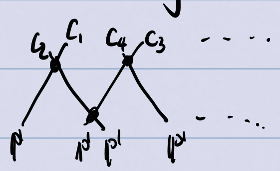
This is singular, with $\mathrm{Sing}(X) =$ image of 0, the fixed pt of the action.

Let $\bar{X}$ be the blowup of X at the sing pt.

$$\begin{tikzcd} \bar{X} \arrow[dr, "p"'] & & \mathbb{A}^2 \arrow[dl] \\ & X & \end{tikzcd}$$

$T^*\mathbb{P}^1$

$n=2$: 

It is well known that $p^{-1}(\mathrm{sing})$ is a union of $\mathbb{P}^1$'s with intersection graph 

Dynkin graph of type A_m

It is not hard to believe that $C_i \cdot C_j = -(\text{Cartan matrix})_{ij}$

Cartan matrix for A_2: $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ ... this gives the condition..

So $(\mathcal{O}_{C_i})$ form an (A_m)-config.

Moreover, it is known that $D^b_G(\mathbb{A}^2) \cong D^b(\tilde{X})$ (*)

If we take $\mathcal{E}_i = \mathcal{O}_0 \otimes V_i \in \mathrm{Coh}_G(X)$, where V_i are the one-dim reps of C_{m+1},

then it is easy to see that

$$\mathrm{Hom}^r_{D^b_G(X)}(\mathcal{E}_i, \mathcal{E}_j) \cong (\Lambda^r R \otimes V_i^\vee \otimes V_j)^G$$

(koszul res for $i_*\mathcal{O}_0$), where $R = \bigoplus_i V_i$, the reg rep.

This also satisfies the conditions.

And in fact $\mathcal{E}_i \longmapsto \mathcal{O}_{C_i}$ under (*)