Nakajima's quiver varieties & Kac-Moody actions with a view toward/from symplectic resolution theory

Main ref: Lectures on Nakajima's quiver varieties by Victor Ginzburg.

https://arxiv.org/pdf/0905.0686.pdf

What do we do:

From Wei's talk, there were 3 things.

1) View things as special cases of Nakajima's quiver varieties, then apply Nakajima's results.

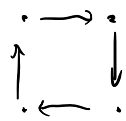
2) Categorify (CKL)

3) Do geometry? (C)

In this talk, we focus on 1), with emphasis on the symp resolution point of view.

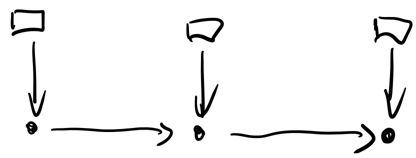
More precisely, we are going to define general Nakajima's quiver varieties and study their (symplectic) geometric properties.

Setup: - quiver = (directed) graph $Q = (I, E)$ (assume without loops (most of the time))

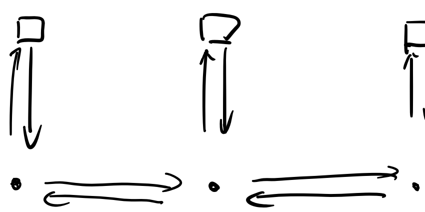
eg 1: 

eg 2: 

- framing



- Take "cotangent space", i.e., double the arrow



Rmk: A few ways of thinking about framing:

1) Nakajima was a differential geometer at one point, studied Gauge theory $\rightsquigarrow$ ADHM equation : $[x, y] + ij = 0$ this $+ ij$ term only appears when you have framing.
2) Thinking quiver varieties as moduli spaces, framing is like "marked points" or "bundles with a choice of trivialisation".

3) (pratical reason), if no framing, the variety is 0 most of the time.

Nakajima quiver variety.

for every vertice $i \in I$, & framing '$\in Q$', chose a number $\mathbb{N}_{\geq 0}$, i.e. $\underline{V}, \underline{W} \in \mathbb{N}^I$. (Think, $\underline{V}, \underline{W}$ as Hilbert polys?)

The space of all reps of the quiver is:

$$\mathrm{Rep}(\overline{Q^\heartsuit}, \underline{V}, \underline{W}) := \bigoplus_{\substack{i \to j \\ j \to i}} \mathrm{Hom}(V_i, V_j) \oplus \bigoplus_{i \to \boxed{i}} \mathrm{Hom}(V_i, W_i)$$

$$\oplus \bigoplus_{i \to \boxed{i}} \mathrm{Hom}(W_i, V_i)$$

where $\dim V_i = v_i$

$\dim W_i = w_i$.

There is a $GL(V) = \bigoplus_{i \in I} GL(V_i)$ action on it,

$$g \cdot (x, y, i, j) = (gxg^{-1}, gyg^{-1}, ig^{-1}, gj)$$

There is G-eqivariant moment map

$$\mu : \mathrm{Rep}(\overline{Q^\heartsuit}, \underline{v}, \underline{w}) \to \mathfrak{g}_v^* \cong \mathfrak{g}_v$$

$$(x, y, i, j) \longmapsto \sum [x, y] + ji \quad \text{(ADHM)}$$

So given $\lambda \in Z(\mathfrak{g}_v)$, $\theta : GL(V) \to \mathbb{C}^\times$

Def: $\mathcal{M}_{\lambda,\theta}(Q, \underline{v}, \underline{w}) := \mu^{-1}(\lambda) /\!\!/_{\theta} GL(V)$

we mostly consider the case $\lambda = 0$.

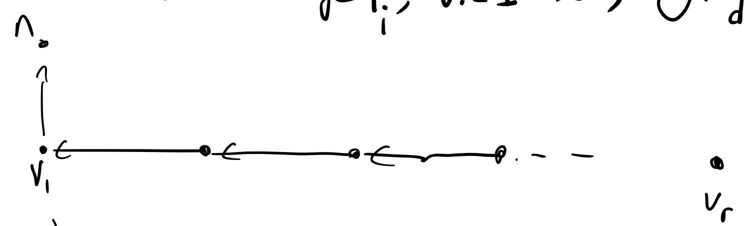
<u>King's stability:</u>

$(x, y, i, j) \in \mu^{-1}(\lambda)$ is θ-semistable iff $\forall\, S_i \subseteq V_i$ which is stable under the maps x & y, we have

$$S_i \subseteq \ker j_i \quad \forall i \in I \Rightarrow \theta \cdot \dim_I S \leq 0$$

$$S_i \supset \mathrm{Image}\, i_i, \quad \forall i \in I \Rightarrow \theta \cdot \dim_I S \leq \theta \cdot \dim_I V$$

<u>Example:</u>



$(n > v_1 > \cdots > v_r)$

$\theta = \theta^+ = (1, \ldots, 1)$

semistable means that x_i & j are injections

$\rightsquigarrow \mathcal{M}_{0,\theta^+} = T^* FL(r, \mathbb{C}^n)$

$\theta = \underline{0} = (0, \cdots, 0)$

Then any pt is θ-semistable.

What is $\mathcal{M}_{0,0}$? (some kind of nilpotent orbit closure ...)

$\theta = \theta^- = (-1, \cdots, -1)$

Semistable means that y_i & j are surjections

$\rightsquigarrow \mathcal{M}_{0,\theta^-} = T^* Fl(r, \mathbb{C}^n)$

but now "flags" are $\mathbb{C}^n \twoheadrightarrow \mathbb{C}^{v_1} \twoheadrightarrow \mathbb{C}^{v_2} \cdots$

$$\begin{tikzcd} \mathcal{M}_{0,\theta^+} \arrow[dr] & & \mathcal{M}_{0,\theta^-} \arrow[dl] \\ & \mathcal{M}_{0,0} & \end{tikzcd}$$

Where is the symp alg geo?

The claim is that $\mathcal{M}_{0,\theta} \to \mathcal{M}_{0,0}$ is an example of a symplectic singularity, & in many cases, a symplectic resolution.

<u>Def</u>: Let X be affine normal Poisson variety.

$\pi: \tilde{X} \to X$ is a symplectic resolution if $\tilde{X}$ is smooth symplectic s.t. $\pi^* \mathcal{O}_X \cong \mathcal{O}_{\tilde{X}}$ as a poisson algebra, and a resolution of singularities.

Quote: 'Symplectic resolutions are the Lie algebras of the 21[st] Century' — Okounkov.

Properties:

1) Semismall: $\dim(\tilde{X} \times_X \tilde{X}) = \dim X$

Therefore dim of irred components $\leq \dim X$

2) X is a union of finitely many symplectic leaves $X = \bigsqcup X_\alpha$, each X_α is locally closed smooth

3) In the case of a conical symplectic resolution (i.e., that there are $\mathbb{C}^\times$ actions on $\tilde{X}$ and X, such that π is equivariant, and contracts X to a point 0

then $\pi^{-1}(0)$ is a homotopy retract of $\tilde{X}$, and $H^\bullet(\tilde{X}, \mathbb{C}) \cong H^\bullet(\pi^{-1}(0), \mathbb{C})$.

4) More generally, π^{-1}(any point) is isotropic (in the sense of symplectic geo)

When is $\mathfrak{M}_{\lambda,\theta}(v,w) \to \mathfrak{M}_{\lambda,0}$ a symplectic resolution?

Answer: (Almost always) when (λ,θ) is v-regular:

$$(\lambda,\theta)\in \mathbb{C}^I\times\mathbb{Z}^I\subseteq \mathbb{C}^I\times\mathbb{R}^I\cong\mathbb{R}^I\times\mathbb{R}^I\times\mathbb{R}^I$$
$$\cong \mathbb{R}^3\otimes\mathbb{R}^I$$

Let $R' := \{\alpha\in\mathbb{Z}^I\setminus\{0\} \mid C_Q v\cdot v\le 2 \quad \forall i\in I\}$

This is the set of roots, when Q is Dynkin or affine Dynkin, this coincides with the usual roots.

C_Q is the cartan matrix, $C_Q := 2I - A_Q$, A_Q is the adjacency matrix.

Back to the example, we had 

$$C_Q = \begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & \ddots & \ddots \\ & & \ddots & 2 \end{pmatrix}$$

and $R' = \{\pm(e_i - e_j)\}$

for $\alpha\in\mathbb{R}^I$, write $\alpha^\perp := \{\lambda\in\mathbb{R}^I \mid \lambda\cdot\alpha = 0\}$

(λ,θ) is v-regular if:

$$(\lambda,\theta)\in(\mathbb{R}^3\otimes\mathbb{R}^I)\setminus\bigcup_{\{\alpha\in R' \mid 0<\alpha\le v\}}\mathbb{R}^3\otimes\alpha^\perp$$

if $(\lambda, \theta) = (0, \theta^+)$, which is $e_1 \otimes 0 \oplus e_2 \otimes 0 \oplus e_3 \otimes \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$
in $\mathbb{R}^3 \otimes \mathbb{R}^I$

$\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \cdot \alpha \neq 0 \implies (0, \theta^+)$ (and $(0, \theta^-)$) is v-regular for all v.

So $\mathcal{M}_{0,\theta^+}(v,w) \longrightarrow \mathcal{M}_{0,0}$
is a symplectic resolution.

(when $\lambda = 0$), the Weyl group $W (= S_n)$ acts on θ's.

& $\mathcal{M}_{0,\theta_1} \cong \mathcal{M}_{0,\theta_2}$ if θ_1, θ_2 in the same chamber.

So, when we were in $\bullet$ (type A_1)
there were 2 chambers $\theta^+ = 1$, $\theta^- = -1$

in $\bullet \!-\!\!-\!\!- \bullet \!-\!\!-\!\!- \bullet - - - \bullet$ type A_n

there are $(1+n)!$ chambers

There is a $\mathbb{C}^*$ action on the cotangent direction:

$$t \cdot (x, y, i, j) = (x, ty, i, tj)$$

& the map $\mathcal{M}_{0,\theta} \to \mathcal{M}_{0,0}$ is $\mathbb{C}^*$-equivariant.

The point is that $\pi^{-1}(\mathcal{M}_{0,0}^{\mathbb{C}^*})$ is a lagrangian subvariety.

and in the case when Q has no oriented cycles, $\mathfrak{M}_{0,0}^{\mathbb{C}^\times} = \{0\}$.

So $\pi^{-1}(0)$ is a Lagrangian in the quiver case.

BM homology

There isn't a notion of fundametal class for non-compact manifolds in usual homology theory, but there is for BM homology.

$$\begin{array}{c} M_1 \times M_2 \times M_3 \\ \big\downarrow p_{ij} \\ M_i \times M_j \\ \cup \text{ closed} \\ Z_{ij} \end{array}$$

$$Z_{12} \circ Z_{23} = p_{13}\left(p_{12}^{-1} Z_{12} \cap p_{23}^{-1} Z_{23}\right)$$

$$*: \; H_i(Z_{12}) \times H_j(Z_{23}) \longrightarrow H_{i+j-\dim M_2}(Z_{12} \circ Z_{23}) \quad (*)$$

$$c_{12} \quad c_{23} \longmapsto p_{13*}\left(\left(c_{12} \boxtimes [M_3]\right) \cap \left(c_{23} \boxtimes M_1\right)\right)$$

Now set $M_i = M$, & $Z = M \times_Y M$ for $\pi: M \to Y$ proper.

This forms an algebra $H_\bullet(Z)$

pick $y \in Y$, $M_y = \pi^{-1}(y)$

Set $M_1 = M_2 = M$, $M_3 = pt$

$Z_{12} = Z$, $Z_{23} = M_y$, $Z_{12} \circ Z_{23} = M_y$

$\rightsquigarrow H_\bullet(Z) \circlearrowright H_\bullet(M_y)$

Now back to the quiver case.

Let $\mathfrak{M}(w) = \bigsqcup_v \mathfrak{M}_{0,\theta^+}(v,w)$

$$\mathfrak{M}_0(w) = \bigsqcup_v \mathfrak{M}_{0,0}(v,w)$$

$$Z(w) = \bigsqcup_{v,v'} \mathfrak{M}_{0,\theta^+}(v,w) \times_{\mathfrak{M}_{0,0}(v+v',w)} \mathfrak{M}_{0,\theta^+}(v',w)$$

(in other words, $Z(w) = \mathfrak{M}(w) \times_{\mathfrak{M}_0(w)} \mathfrak{M}(w)$)

Let $H_w = H_{top}(Z(w))$

Let $\pi^{-1}_{v,w}(0)$ be the Lagrangian
$$\begin{array}{c} \mathfrak{M}_{0,\theta^+}(v,w) \\ \downarrow \pi_{v,w} \\ \mathfrak{M}_{0,0} \end{array}$$

$$L_w = H_{top}\left(\bigsqcup_v \pi^{-1}_{v,w}(0)\right)$$

Using top as there is a shift in (☆), and semismall property makes sure we stay in top deg. And Lagrangian also has the right dim.

(I think)

$\rightsquigarrow \quad H_w \circlearrowright L_w$

Theorem [Na]: There is an algebra map

$$\Phi : \tilde{U}(\mathfrak{g}_Q) \longrightarrow H_w,$$

and L_w is a simple integrable $\mathfrak{g}_Q$-module with highest weight $\sum_{i \in I} w_i \cdot \omega_i$ (ω_i fundamental weight)

When Q is type A, this was first discovered by Ginzburg, "Lagrangain construction of the evaloping algebra $U(\mathfrak{sl}_n)$"

Define $B_k^{(r)}(v, w) = \{ (V', V'') \mid V'' \in \mathrm{Rep}(\bar{Q}, v + re_k, w),$
$V' \subset V''$ subrep
s.t. $\mathrm{Im}(i_k : W_k \to V''_k) \subset V' \}$

$B_k^{(r)}(v, w)$ is a irreducible component in $Z(v, v+re_k, w)$

Define $E_k^{(r)} = \sum_v [B_k^{(r)}(v, w)]$

Let $\Delta(v, w)$ be the diagonal in $\mathfrak{M}_{0,\theta}(v, w) \times \mathfrak{M}_{0,\theta^+}(v, w)$

Then $E_k [\Delta(v, w)] = [\Delta(v - e^k, w)] E_k$

Appearnely this is easy to check..