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D-modules on singular varieties and Hochschild homology of quantisations

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I certify that this thesis, and the research to which it refers, are the product of my own work, and that any ideas or quotations from the work of other people, published or otherwise, are fully acknowledged in accordance with the standard referencing practices of the discipline.

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Abstract

In the first part of the thesis, we study the category of D-modules on a (singular) variety. We show for an affine variety X, the derived category of quasi-coherent D-modules is equivalent to the category of DG modules over an explicit DG algebra, whose zeroth cohomology is the ring of Grothendieck differential operators $\mathrm{Diff}(X)$. When the variety is cuspidal, we show that this is just the usual ring $\mathrm{Diff}(X)$, and the equivalence is the abelian equivalence constructed by Ben-Zvi and Nevins. We compute the cohomology algebra and its natural modules in the hypersurface, curve and isolated quotient singularity cases. We identify cases where a D-module is realised as an ordinary module (in degree 0) over $\mathrm{Diff}(X)$ and where it is not.

In the second part, we define an analogue for Hochschild homology of a construction of Etingof–Schedler which enhances the (zeroth) Poisson homology to a local version, defined using a specific D-module $M(X)$. This uses another D-module $M_\hbar(X)$ and yields a new version of Hochschild homology with desirable features, called *Hochschild–de Rham* homology. In general, the Hochschild–de Rham homology agrees with the ordinary Hochschild homology in degree 0 when X is affine. We study in detail the case of certain symplectic resolutions and show that Poisson-de Rham homology and the Hochschild–de Rham homology agree with the de Rham cohomology of the symplectic resolution. We show that if X has finitely many symplectic leaves, then $M_\hbar(X)$ is in a sense holonomic and hence deduce a finite generation result about Hochschild–de Rham homology. Finally, in the smooth setting, we conjecture that the Hochschild–de Rham homology of the canonical Kontsevich quantisation of the Poisson structure is isomorphic to the Poisson-de Rham homology of X.

In the third part, we apply the results of the second part to skein theory of tori. We find an explicit symplectic resolution $\mathrm{Hilb}^0(T^2)$ of the SL_N-character variety of the 2-torus T^2, which will be a version of a Hilbert scheme of T^2. We show that $\mathrm{Hilb}^0(T^2) \times T^2$ is a $C_N \times C_N$ covering of $\mathrm{Hilb}(T^2)$ and hence compute its de Rham cohomology and the Hochschild–de Rham homology of quantisations of it. The Skein module is a quantisation of the character variety, and we deduce that $\dim \mathrm{Sk}_{SL_N}(T^3) = \mathcal{P} \star J_3(N)$, where $\mathcal{P}$ is the number of partitions function, J_3 is the third Jordan function and $\star$ is Dirichlet convolution.

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For Zhen

Contents

Chapter 1

Introduction

1.1 D-modules on singular varieties

Suppose X is a smooth complex affine variety. Then D-modules on X are defined to be modules over the ring of Grothendieck differential operators Diff(X) which behaves nicely. It is well known that Diff(X) is Noetherian in this case. If X is not affine but still smooth, we can sheafify this construction to obtain a sheaf $\mathfrak{Diff}_X$, and define the D-modules as the sheaves of modules over this sheaf of rings.

For smooth varieties, leads to a theory with many desirable properties. However, for singular varieties many problems can occur. The fundamental issue is that the ring of differential operators can be very complicated and sometimes not even Noetherian; a non-Noetherian example is the cubic cone $x_1^3 + x_2^3 + x_3^3 = 0$, see [BGG72]. Even when it is Noetherian, in general this construction does not have desirable geometric properties. For instance, Kashiwara's theorem [HTT08, Theorem 1.6.1], that for a closed embedding $X \hookrightarrow Y$, $\mathfrak{Diff}_Y$-modules set-theoretically supported on X are equivalent to $\mathfrak{Diff}_X$-modules, fails in general for any non-cuspidal X and Y (see Section 2.1.2).

A typical solution is to *define* the category so that this statement holds, *i.e.,* choose a closed smooth embedding $X \hookrightarrow Y$ and define D_Y-mod$_X$ to be the category of D-modules on Y set-theoretically supported on X are equivalent to D-modules on X. It can be shown that D_Y-mod$_X$ is independent of the choice of Y (see Section 2.1.1), hence we may abbreviate the notation to D-mod$_X$ if we want to talk about this category without the reference to a chosen closed embedding. But then, this is no longer the category of modules over any ring. This leads to two definitions of D-modules ($\mathfrak{Diff}_X$-mod and D-mod$_X$). Another one, called *crystals* Crys$^r(X)$ were first introduced by Grothendieck in [Gro68] where he defined *crystalline topos* and related them to de Rham theory; Beilinson and Drinfeld in [BD] related crystals to D-

modules and noted that the embedding theorem holds automatically in both smooth and non-smooth settings, and in [GR14] Gaitsgory and Rozenblyum gave a modern treatment. Crystals are defined as sheaves on X which are equipped with compatible extensions (analogously to parallel transport) on infinitesimal thickenings. These three definitions coincide for smooth varieties, that is

$$\mathfrak{Diff}_X\text{-mod} \cong D\text{-mod}_X \cong \text{Crys}^r(X)$$

(depending on which definition of crystals we use, this might need to be upgraded to a derived equivalence. Modern definitions of crystals only make sense in the derived/infinity category setting). More generally, it is shown in [SS88] and generalised in [BN04], when the variety is Cohen–Macaulay and a cuspidal curve (or more generally there is a *good cuspidal quotient morphism* from a smooth variety to it), these three definitions coincide.

One approach to singular varieties is via '*derived algebraic geometry*' - roughly, this replaces ordinary rings by DG rings, and categories by DG (or triangulated) categories. In this context, it turns out that all three definitions, suitably interpreted, coincide, and give a triangulated category of D-modules $D(D\text{-mod}_X)$.

In this work, we consider the relationship between this DG category and the original viewpoint of rings of Grothendieck differential operators. We prove that the derived category of D-modules is equivalent to the category of DG modules over an explicit DG algebra $\mathfrak{Diff}_X^{dg}$, which 'corrects' Grothendieck's ring. Namely, it is concentrated in non-negative degrees, with zeroth cohomology equal to $\mathfrak{Diff}_X$, and with cohomology bounded by the dimension of the variety (so it is a nilpotent extension of the original ring).

The starting point of this work is the observation that there is a D-module $D_X \in D\text{-mod}_X$ on (singular) varieties which is a compact generator of the $D(D\text{-mod}_X)$. The fact that D_X is a compact generator was already proved in [GR14] in the framework of crystals, but we give an elementary proof in terms of modules over rings of differential operators.

We easily deduce from the compact generator D_X that:

Theorem 1.1.1. There is a derived equivalence between the derived categories of D-modules on X and DG modules over $\mathfrak{Diff}_X^{dg}$,

$$D(D\text{-mod}_X) \cong D(\mathfrak{Diff}_X^{dg}\text{-mod}).$$

We emphasise that the result Theorem 1.1.1 itself is an easy consequence of [GR14], but the contribution of this work is to elucidate the structure of $\mathfrak{Diff}_X^{dg}$ and explore the consequences of this statement.

In particular, in the case X is smooth or more generally cuspidal, we show $\mathfrak{Diff}_X^{dg} \cong \mathfrak{Diff}_X$ is concentrated in degree zero. And we recover the fact that, in cuspidal/smooth cases, all definitions of D-modules coincide even in the underived setting. When X need not be cuspidal, the idea to study D-modules as DG modules over a DG algebra is new, to our knowledge.

The structure of chapter 2 of the thesis is as follows:

In Section 2.1, we prove Theorem 1.1.1. We write down the explicit compact generator $D_X \in D(D\text{-mod}_X)$ in the affine case. We give an elementary proof that it compactly generates the derived category of D-modules on V supported on X, for any closed embedding of X into a smooth affine variety V. We also explain how our construction agrees with [BN04] in the cuspidal case.

We recall the seminormalisation X^{sn} and explain that:

Theorem 1.1.2. There is a derived equivalence between the categories

$$D(D\text{-mod}_X) \cong D(D\text{-mod}_{X^{sn}}).$$

Note that X is cuspidal if and only if X^{sn} is smooth. Moreover, if X is Cohen–Macaulay, the equivalence is the derived version of the one explained in [BN04] (and our argument in general is essentially the same as theirs).

In Section 2.2, in the case of a hypersurface, from Theorem 1.1.1 we derive the interesting identity:

Corollary 1.1.3. If $X = \{f = 0\}$ is a cuspidal hypersurface, then

$$\frac{D_{\mathbb{A}^n}}{D_{\mathbb{A}^n} \cdot f + f \cdot D_{\mathbb{A}^n}} = 0.$$

The formula computes $\operatorname{Ext}^1(D_X, D_X)$, and it vanishes if and only if $\operatorname{Diff}(X) = \operatorname{Diff}(X)^{dg}$; that is, if D-modules on X are the same as $\operatorname{Diff}(X)$-modules. For general f, we compute $H^\bullet(\operatorname{Diff}(X)^{dg}) = \operatorname{Ext}^\bullet(D_X, D_X)$ and its action on D-modules (more precisely, on $\operatorname{Ext}^\bullet(D_X, M)$).

Note that if $\operatorname{Ext}^m(D_X, M)$ for all $m > 0$, then under Theorem 1.1.1, M can be realised as an ordinary D-module over $\operatorname{Diff}(X)$.

In Section 2.3, we give some examples of our formulas and theorems in the case of regular holonomic D-modules when the variety is a curve C. Define completely nontrivial monodromy to mean that in the normalisation of C, all eigenvalues of monodromies about exceptional points are not equal to 1 (see Definition 2.3.6). By calculating $\mathrm{Ext}^1(D_C, M)$, we show:

Theorem 1.1.4. The abelian subcategory of regular holonomic D-modules with completely non-trivial monodromy around each non-cuspidal singularity over a curve can be realised as ordinary modules over $\mathrm{Diff}(C)$, *i.e.*, $\mathrm{Ext}^m(D_C, M)$ for all $m > 0$ and M in this abelian subcategory.

The converse also holds for simple regular holonomic D-modules on seminormal curves.

Finally in Section 2.4, we study the case of holonomic D-modules on isolated quotient singularities. In this case the results actually have the opposite flavour:

Theorem 1.1.5. For an isolated finite quotient singularity X, let L be a $\mathbb{C}^*$-equivariant local system, if there is trivial monodromy about singularities then L can be realised as ordinary modules over $\mathrm{Diff}(X)$. The converse holds for intersection cohomology D-modules.

In [HK84] and later in [ES09] (see also [ES17]) a certain quotient $M(X)$ of our compact generator D_X was considered which governs the invariants under Hamiltonian flows. It was used to define a new homology theory which fuses Poisson homology with the de Rham cohomology, which is particularly nice in the case of symplectic singularities. Other quotients of D_X were studied in [ES12], relating to other geometric structures on X. We hope that our study will have applications to these quotients and plan to address this elsewhere.

1.2 Hochschild homology of quantisations

Let X be an affine Poisson variety (not necessarily smooth) over an algebraically closed field of characteristic 0 (such as $\mathbb{C}$). Denote $\mathcal{O}(X)$ its ring of functions and $\{-,-\}$ its Poisson structure. A *star product* $\star$ on $\mathcal{O}(X)[\![\hbar]\!]$ is a $\mathbb{C}[\![\hbar]\!]$-bilinear associative unital (with unit the constant function 1) map $\mathcal{O}(X)[\![\hbar]\!] \times \mathcal{O}(X)[\![\hbar]\!] \to \mathcal{O}(X)[\![\hbar]\!]$, such that $f \star g = fg \mod \hbar$. Therefore there is a sequence of maps

$\phi_i : \mathcal{O}(X) \times \mathcal{O}(X) \to \mathcal{O}(X)$ such that

$$f \star g = fg + \sum_{i \geq 1} \hbar^i \phi_i(f, g) \in \mathcal{O}(X)[\![\hbar]\!].$$

Furthermore, if each ϕ_i is a bi-differential operator, we call $\star$ a *differential star product*. Note that, if X is smooth, then it is shown in [Yek13, Theorem 8.2] that every star product is gauge equivalent to a differential star product. It is an interesting question if there exists a similar statement in the non-smooth case to the aforementioned result of Yekutieli. In chapter 3 of the thesis, we assume all of our star products are differential star products.

A *deformation quantisation* of $(\mathcal{O}(X), \{-,-\})$ is by definition $\mathcal{O}(X)[\![\hbar]\!]$ with a star product such that $\phi_1(f,g) - \phi_1(g,f) = \{f,g\}$. We denote it $\mathcal{O}_\hbar(X) = (\mathcal{O}(X)[\![\hbar]\!], \star)$.

In order to study the Poisson structure $\{-,-\}$ (equivalently $\pi \in \Gamma(X, \bigwedge^2 TX)$) (resp. its deformation), it is natural to consider the Poisson homology (resp. Hochschild homology). Let us briefly recall the definition and some of the features of Poisson homology and Hochschild homology here.

Poisson homology can be defined in at least two ways: one using the cotangent complex $\mathbb{L}_X$ with the differential $d_{\text{Poiss}} := L_\pi = [d_{dR}, i_\pi]$ and one using a double complex. The latter is closely related to the Hochschild complex $HC_\bullet(A,A) := (T_k^{\geq 1} A, d_{\text{Hoch}})$. The zeroth Poisson homology $\mathbf{HP}_0(X, \pi)$ is given by $\mathcal{O}(X)/\{\mathcal{O}(X), \mathcal{O}(X)\}$, which is the vector space dual to the *Poisson traces*

$$\{f : \mathcal{O}(X) \to \mathbb{C} | f(\{a,b\}) = 0 \text{ for all } a, b \in \mathcal{O}(X)\}.$$

The zeroth Hochschild homology $\mathbf{HH}_0(\mathcal{O}(X)[\![\hbar]\!])$ is given by $\mathcal{O}_\hbar(X)/[\mathcal{O}_\hbar(X), \mathcal{O}_\hbar(X)]$. Higher homologies have less clear interpretations.

One may also ask about Poisson cohomology or Hochschild cohomology. In certain cases, for example the unimodular case, there is a duality between Poisson homology and cohomology, and in the Calabi–Yau case, between Hochschild homology and cohomology. We focus on the homology theory in this thesis.

Although the zeroth Poisson homology of X is globally defined, it can be obtained from a local object. In particular, in [ES09, Definition 2.2] an explicit D-module on X was defined which encodes the zeroth Poisson homology. This is the main idea in their paper. We recall the definition here:

Definition 1.2.1. $M(X) := (\text{Ham}_X) \backslash D_X$, where the submodule (Ham_X) is the (right) submodule of D_X spanned by Hamiltonian vector fields ξ_f, where for $f \in \mathcal{O}(X)$, $\xi_f \in \mathrm{Vect}(\mathcal{O}(X))$ is defined by

$$\xi_f(g) = \{f, g\}.$$

Here and below, $N\backslash M$ denotes the (right) quotient module for N a submodule of M. We use right modules because they behave more naturally with respect to the D-module-theoretic pushforward, which we will need later, see below.

In [ES09], the authors used this D-module to define the so called *Poisson-de Rham* homology of X:

$$\mathbf{HP}_i^{dR}(\mathcal{O}(X)) := H^{-i}\pi_* M_X,$$

where π_* is the D-module theoretic derived pushforward from X to a point.[1] It was shown in their paper that

$$\mathbf{HP}_0(\mathcal{O}(X)) = \mathbf{HP}_0^{dR}(\mathcal{O}(X)).$$

Both the definition of the D-module and Poisson-de Rham homology make sense for non-affine X.

The functor of point perspective for the above constructions is the following: D_X is the object that represents the functor of global sections on the category of right D-modules on X; that is, if we fix a closed embedding $X \to V$ where V is affine and smooth, for a D-module N set-theoretically supported on X,

$$\Gamma_X(N) = \mathrm{Hom}(D_X, N),$$

where $\Gamma_X(N)$ is the subspace of the global sections of N as a sheaf on V which are scheme-theoretically supported on X, i.e., locally annihilated by the ideal sheaf of X. It is also possible to write down the functor of point perspective without assuming that X is affine, see [ES17, Section A.2].

$M(X)$ is the object that represents the functor of Hamiltonian invariant global sections on the category of right D-modules on X; that is,

$$\Gamma_X(N)^{(\mathrm{Ham}_X)} = \mathrm{Hom}(M(X), N),$$

where $\Gamma_X(N)^{(\mathrm{Ham}_X)}$ denotes the sections that are annihilated by the ideal (Ham_X) which can be seen as a Lie algebra.

[1] In [ES09] they used the notation $\mathbf{HP}_i^{dR}(X)$, but here we prefer $\mathbf{HP}_i^{dR}(\mathcal{O}(X))$ notation to be consistent with $\mathbf{HH}_i^{dR}(\mathcal{O}_\hbar(X))$, defined in Section 3.1.

In chapter 3 of the thesis, we generalise the construction $M(X)$ for an affine Poisson variety X to $M_\hbar(X)$ for a quantisation $\mathcal{O}_\hbar(X)$ of $\mathcal{O}(X)$. The module $M_\hbar(X)$ represents quantised Hamiltonian ($\mathrm{Ham}_{\hbar,X}$) (see chapter 3) invariant global sections; that is, if we fix a closed embedding $X \to V$ where V is affine and smooth, for a $D_V[\![\hbar]\!]$ module N set-theoretically supported on X (ignoring the $\hbar$ action),

$$\Gamma_X(N)^{(\mathrm{Ham}_{\hbar,X})} = \mathrm{Hom}(M_\hbar(X), N),$$

where Hom now is taken over the category of $D_V[\![\hbar]\!]$-modules.

Following the idea defining the Poisson-de Rham homology in [ES09], we use $M_\hbar(X)$ to define *Hochschild–de Rham* homology $\mathbf{HH}_i^{dR}(\mathcal{O}_\hbar(X))$ of $\mathcal{O}_\hbar(X)$. This gives a local enhancement of Hochschild homology of quantisations. Hochschild–de Rham homology has nice features such as it is bounded and well-behaved for quantisations of symplectic singularities.

The construction in chapter 3 of the thesis can also be extended to non-affine varieties simply by gluing these $M_\hbar(X)$ together for a sheaf of quantisations.

Chapter 3 is structured as follows:

In Section 3.1, we define $M_\hbar(X)$ and use it to define the Hochschild–de Rham homology of a quantisation. We show it agrees with the usual Hochschild homology in degree 0. We prove there is a canonical surjection $M(X)[\hbar] \twoheadrightarrow \mathrm{gr}_\hbar M_\hbar(X)$ of $D_X[\hbar]$-modules. We also study the case when X is smooth symplectic. Then the canonical surjection becomes an isomorphism and hence we deduce that the Hochschild–de Rham homology agrees with the usual Hochschild homology in this case.

In Section 3.2, we study $M_\hbar(X)$ in the presence of symplectic resolution $\rho : \tilde{X} \to X$, generalising the smooth symplectic case. Our main result of the paper is the following theorem.

Theorem 1.2.2. Let $\rho : \tilde{X} \twoheadrightarrow X$ be a projective symplectic resolution such that

- $\rho_* \Omega_{\tilde{X}} \cong M(X)$,
- X has locally conical singularities.

Assume further that the quantisation $\mathcal{O}_\hbar(X)$ extends to a quantisation $\mathcal{O}_\hbar(\mathfrak{X})$ on a (one-parameter) smoothing $\mathfrak{X}$ of X. Then $M(X)[\hbar] \cong \mathrm{gr}\, M_\hbar(X)$ as graded modules. Moreover this can be strengthened to $M_h(X) \to M(X)[\![\hbar]\!] \cong \rho_*\Omega_{\tilde{X}}[\![\hbar]\!]$ is an isomorphism as filtered modules.

Let X and $\mathcal{O}_\hbar(X)$ satisfy the conditions of the theorem above. In particular, we see that in most cases:

Corollary 1.2.3.

$$\mathbf{HH}_0(\mathcal{O}_\hbar(X)) \cong \mathcal{O}(X) \oplus H^{\dim X}(\tilde{X}, \mathbb{C}[\![\hbar]\!])$$

and

$$\mathbf{HP}_0(\mathcal{O}_\hbar(X))(\!(\hbar)\!) \cong \mathbf{HH}_0(\mathcal{O}_\hbar(X)[\hbar^{-1}]) \cong H^{\dim X}(\tilde{X}, \mathbb{C}(\!(\hbar)\!))$$

and it is independent of the quantisation, generalising the Nest–Tsygan theorem.

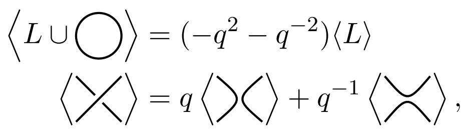
In Section 3.3, we study the holonomicity of $M_\hbar(X)$. More specifically we prove that $M_\hbar(X)/\hbar^n M_\hbar(X)$ is holonomic for all n if X has finitely many symplectic leaves and hence deduce finite generation of Hochschild homology in certain cases.

Finally, in Section 3.4, we study the smooth case and conjecture an isomorphism between the Hochschild–de Rham homology of the canonical Kontsevich quantisation of the Poisson structure and the Poisson-de Rham homology of X.

1.3 Quantum topology and Skein theory

A fundamental invariant of an oriented 3-manifold M from quantum topology is its "Kauffman bracket skein module" $\mathrm{Sk}(M)$ introduced by Józef Przytycki and Vladimir Turaev. We recall the definition here.

Definition 1.3.1. For an oriented manifold M, $\mathrm{Sk}(M)$ is defined as the $\mathbb{C}[q, q^{-1}]$-module formally spanned by all framed links in M, modulo isotopy equivalence and the linear "Kauffman bracket" relations



which are imposed between any links agreeing outside of an oriented 3-ball, and differing as depicted inside that ball.

In fact, for every ribbon category $\mathcal{A}$, it is possible to define a vector space $\mathrm{Sk}_{\mathcal{A}}(M)$. When $\mathcal{A} = \mathrm{Rep}_q(G)$, for a reductive group G, we abbreviate it to $\mathrm{Sk}_G(M)$. When $G = SL_2(\mathbb{C})$, we recover the definition above. It is possible to give a diagrammatic description of G-skein modules for other groups analogous to the Kauffman skein relations, though it becomes more complicated. See [CKM14] for a description for SL_n.

It was conjectured by Edward Witten that for a closed manifold M, the skein module

$\mathrm{Sk}_G(M)$ is finite dimensional over $\mathbb{C}(q)$. This was proved by Sam Gunningham, David Jordan and Pavel Safronov recently in [GJS19]. However, the proof they gave is not by directly computing the dimensions. Rather, it is one of a number of consequences of their theorems, which gives a new algebraic reformulation of skein modules, and brings tools from the representation theory of quantum groups and deformation quantization modules.

We explore some of their main ideas and compute explicitly the dimension of the SL_n-skein module $\mathrm{Sk}_{SL_n}(T^3)$ of the three dimensional torus T^3 and show that

$$\dim \mathrm{Sk}_{SL_n}(T^3) = \mathcal{P} \star J_3(n),$$

where $\mathcal{P}$ is the number of partitions function, J_3 is the third Jordan function and $\star$ is the Dirichlet convolution. This generalises the results of [Car17] and [Gil16], which showed that $\dim \mathrm{Sk}_{SL_2}(T^3) = 9$.

Let's recall some further basic notions in skein theory. If Σ is an oriented surface, the skein module $\mathrm{Sk}_{\mathcal{A}}(\Sigma \times [0,1]) = \mathrm{SkAlg}_{\mathcal{A}}(\Sigma)$ is a skein algebra, where the composition is given by stacking skeins on top of each other. Similarly, if M is a 3-manifold with boundary Σ, then $\mathrm{Sk}_{\mathcal{A}}(M)$ is naturally a module over $\mathrm{SkAlg}_{\mathcal{A}}(\Sigma)$.

One can use the above to construct a TFT $Z_{\mathcal{A}}$ due to Kevin Walker. Namely, the assignment of a skein module $\mathrm{Sk}_{\mathcal{A}}(M)$ to a closed 3-manifold M and a *skein category* $\mathrm{SkCat}_{\mathcal{A}}(\Sigma)$ to a closed 2-manifold Σ is a part of a topological field theory valued in categories and their bimodules. See [GJS19, Chapter 2] for the definition of skein category and relevant details.

It is a general feature of topological field theories that the value on $S^1 \times X$ yields the corresponding categorical dimension of the value on X. For a vector space, the categorical dimension is the ordinary dimension (an integer) while for a category, it is the categorical trace, or zeroth Hochschild homology. In [GJS19, Lemma 4.5], it is proved that $\mathrm{Sk}_{\mathcal{A}}(S^1 \times \Sigma) \cong \mathbf{HH}_0(\mathrm{SkCat}_{\mathcal{A}}(\Sigma))$ (or equivalently $\mathbf{HH}_0(Z_{\mathcal{A}}(\Sigma))$).

Disclaimer: The following (and chapter 4) will appear as part of a joint work [Gun+] between the author and Sam Gunningham, David Jordan and Monica Vazirani.

Fix $G = SL_n$, let $\mathbb{T} := (\mathbb{C}^*)^{n-1}$ be a maximal torus of SL_n and let W be the Weyl group. For the case of the torus $T^3 = T^2 \times S^1$ (not to be confused with $\mathbb{T}$), in [GJS19, Theorem 3], the following decomposition of abelian categories was advertised

$$Z_{SL_n}(T^2) \cong \mathrm{LMod}_{D_q(\mathbb{T})^W} \bigoplus \mathrm{Vect}^{\oplus k},$$

where $W = S_n$ is acting on the quantum $(n-1)$-torus $D_q(\mathbb{T})$ multiplicatively. Furthermore, the following isomorphism of algebras was desired

$$D_q(\mathbb{T})^W \cong \mathrm{SkAlg}_{SL_n}(T^2)$$

(see [FG98] for the SL_2 case). Gunningham, Jordan and Vazirani proved both isomorphisms in our collaboration. The algebra $D_q(T)^W$ is known as the algebra of W-invariant q-difference operators (see [BBJ18, Section 1.5]). Therefore,

$$\mathrm{Sk}_{SL_n}(T^3) \cong \mathbf{HH}_0(D_q(\mathbb{T})^W) \bigoplus \mathbb{C}^k.$$

If we know the dimension of $\mathbf{HH}_0(D_q(\mathbb{T})^W)$, then it is possible to compute the dimension of the entire skein module $\mathrm{Sk}_{SL_n}(T^3)$. Indeed, for any 3-manifold M, the mapping class group of M acts on the skein module $\mathrm{Sk}_G(M)$. When $M = T^3$, it is well-known that the mapping class group is $SL_3(\mathbb{Z})$ (see, for example, [HW07], though earlier reference must exist). Furthermore, there is also a $H_1(M, \mathbb{Z}/n\mathbb{Z})$ grading for $\mathrm{Sk}_{SL_n}(M)$, and in the case $M = T^3$, the grading group is $(\mathbb{Z}/n\mathbb{Z})^3$. The mapping class group acts on the skein module compatibly with the grading group. In the case $M = T^3$, this is via a reduction homomorphism $SL_3(\mathbb{Z}) \to SL_3(\mathbb{Z}/n\mathbb{Z})$. With this $(\mathbb{Z}/n\mathbb{Z})^3$ grading, we can show $\mathbf{HH}_0(D_q(\mathbb{T})^W)$ is none other than the sum of the components in degree $(a, b, 0)$ for this $(\mathbb{Z}/n\mathbb{Z})^3$ grading. But then this is all one needs to do, because the hyperplane of the form $(a, b, 0)$ in the grading group generates all degrees (a, b, c) under the $SL_3(\mathbb{Z}/n\mathbb{Z})$ action. See [Gun+] for more detail.

My main contribution to the collaboration is the computation of $\mathbf{HH}_0(D_q(\mathbb{T})^W)$. We solely focus on this part of the collaboration in this thesis.

It is well-known that $\mathrm{SkAlg}_{SL_n}(T^2)$ is a quantisation of the SL_n-character variety (see [Tur91])

$$\mathrm{Hom}(\pi_1(T^2), SL_n)///SL_n = (SL_n \times SL_n)///SL_n \cong (\mathbb{C}^*)^{2(n-1)}/S_n,$$

where the last isomorphism follows from [BS21a, Proposition 2.8]. This is a singular symplectic variety when equipped with the Atiyah–Bott–Goldman structure. This in fact admits a symplectic resolution (see chapter 4). It is known that $D_q(\mathbb{T})^W$ has a one-parameter deformation $H(W)$, the *spherical double affine Hecke algebra* (sDAHA). Again, see [BBJ18, Section 1.5].

We show the machinery of Chapter 3 will apply in this case. We hope to extend this calculation to compute $\mathrm{Sk}_{SL_n}(\Sigma_g \times S^1)$.

1.4 Convention and recollections of D-modules

By a variety X, we always mean a reduced separated scheme of finite type over $\mathbb{C}$. We will always work with quasi-coherent D-modules.

We use the straight D_X for the explicit D-module we define in this paper. We use $\mathfrak{Diff}_X$ for the sheaf of rings of differential operators on X (which coincides with D_X when X is smooth). When X is affine, we reserve the notation $\mathrm{Diff}(X)$ for the ring of Grothendieck differential operators on X. **By a D-module on X, we will always mean an element of D-mod$_X$.** We will explicitly say $\mathrm{Diff}(X)$-module when we mean it. They of course coincide when X is smooth.

We use the term local system to mean an $\mathcal{O}$-coherent right D-module (equivalently, a vector bundle with a flat connection) on a locally closed smooth subvariety. We use the term topological local system to mean a representation of the fundamental group of such a subvariety. The Riemann–Hilbert correspondence give an equivalence of categories of topological local systems and local systems with regular singularities. We write $\mathbf{IC}(X)$ for the intermediate extension of the trivial local system.

In most scenarios, X will also be affine and it will have an embedding into $\mathbb{A}^n$.

We also recall some aspects of the **six functor formalism** on the derived category of D-modules, which we will use throughout the thesis. Here, all functors are taken to be derived.

Let $f : X \to Y$ be a morphism of smooth irreducible algebraic varieties, $N \in D(D_Y\text{-mod})$ be a left module, $M \in D(D_X\text{-mod})$ be a left module and $d = \dim X - \dim Y$.

- The functor $f^!$ is defined on the derived category of all D-modules:

$$f^!(N) := f^\bullet(N)[d],$$

  where $f^\bullet$ is the derived quasi-coherent pullback. The D-module structure is defined via the product rule, [HTT08, Page 33].

- The transfer bimodules are:

$$D_{X\to Y} := f^\bullet(D_Y) = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_X} f^{-1}D_Y,$$

$$D_{X\leftarrow Y} := \Omega_X \otimes_{\mathcal{O}_X} D_{X\to Y} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\Omega_Y^{\otimes -1},$$

  where Ω is the canonical sheaf naturally viewed as a right D-module and f^{-1} is the sheaf-theoretic pullback, [HTT08, Definition 1.3.1, 1.3.3].

- The functor f_* is defined on the derived category of all D-modules:

$$f_*(M) := f_\bullet(D_{Y\leftarrow X} \otimes^{\mathbb{L}}_{D_X} M),$$

where $f_\bullet$ is the derived quasi-coherent pushforward (which coincides with the abelian sheaf theoretic pushforward), [HTT08, Page 40].

- The functors f_* and $f^!$ are compatible with compositions, [HTT08, Proposition 1.5.11, 1.5.21].

- Let $i : Z \to X$ be a closed embedding, and $j : X\backslash Z \to X$ the corresponding open embedding, then we have an exact triangle:

$$i_*i^!M \to M \to j_*j^!M \to,$$

[HTT08, Proposition 1.7.1].

- The functors f_* and $f^!$ are compatible with base change. That is, if

$$\begin{array}{ccc} X \times_Y S & \xrightarrow{\tilde{g}} & X \\ {\scriptstyle \tilde{f}}\downarrow & & \downarrow{\scriptstyle f} \\ S & \xrightarrow[g]{} & Y \end{array}$$

is a pullback diagram, then:

$$g^! \circ f_* = \tilde{f}_* \circ \tilde{g}^!,$$

[HTT08, Theorem 1.7.3].

- The functor $\mathbb{D}$ is defined on the derived category of *coherent* D-modules, and mapping to the opposite category:

$$\mathbb{D}(M) := \mathrm{Hom}_{D_X}(M, D_X) \otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1}[\dim X].$$

Moreover, $\mathbb{D}^2 = \mathrm{Id}$. [HTT08, Proposition 2.65].

- The functor $f_! := \mathbb{D} f_* \mathbb{D}$ is defined on the *coherent* D-modules M such that $f_*\mathbb{D}(M)$ is coherent. [HTT08, Definition 3.2.12], but it could be defined earlier in this reference.

- The functor $f^* := \mathbb{D} f^! \mathbb{D}$ is defined on the *coherent* D-modules M such that $f^!\mathbb{D}(M)$ is coherent. [HTT08, Definition 3.2.12], but it could be defined earlier in this reference.

- If f is *proper*, then f_* preserves coherence, [HTT08, Theorem 2.5.1], and

  $$f_* = f_!,$$

  [HTT08, Theorem 2.7.2]. Moreover $f_!$ is left adjoint to $f^!$ on the derived categories of coherent D-modules,

  $$f_! \dashv f^!,$$

  [HTT08, Corollary 2.7.3].

- If f is a *closed embedding*, then f_* is exact. This follows from Kashiwara's equivalence [HTT08, Theorem 1.6.1].

- If f is *smooth*, then $f^!$ preserves coherence, [HTT08, Proposition 1.5.13 (ii)] and

  $$f^* = f^![-2d],$$

  [HTT08, Theorem 2.7.1].

So when f is *étale*, in particular an *open embedding*, $f^* = f^!$.

Also $f^![-d]$ is exact, [HTT08, Theorem 2.4.6 (i)].

Moreover, f_* is right adjoint to f^* on the derived categories of coherent D-modules,

$$f^* \dashv f_*.$$

This fact is not proved in [HTT08] and is mentioned in [Bra+, Theorem 2.3.9] without proof. We don't need to use the full version of the result except the case when we have an open embedding. However, we include a proof for completeness of the thesis. We will use the fact $f^* = f^![-2d]$ and our proof is very similar to [HTT08, Corollary 2.7.3].

Proof. We want to prove

$$f_\bullet \operatorname{RHom}_{D_X}(f^!M[-2d], N) \cong \operatorname{RHom}_{D_Y}(f_*M, N).$$

We have

$$\begin{aligned}
f_\bullet \operatorname{RHom}_{D_X}(f^! M[-2d], N) &\cong f_\bullet((\Omega_X \otimes^{\mathbb{L}}_{\mathcal{O}_X} \mathbb{D} f^! M[-2d]) \otimes^{\mathbb{L}}_{D_X} N)[-\dim X] \\
&\cong f_\bullet((\Omega_X \otimes^{\mathbb{L}}_{\mathcal{O}_X} f^! \mathbb{D} M) \otimes^{\mathbb{L}}_{D_X} N)[-\dim X] \\
&\cong f_\bullet((\Omega_X \otimes^{\mathbb{L}}_{\mathcal{O}_X} N) \otimes^{\mathbb{L}}_{D_X} f^! \mathbb{D} M[-\dim X]) \\
&\cong f_\bullet((\Omega_X \otimes^{\mathbb{L}}_{\mathcal{O}_X} N) \otimes^{\mathbb{L}}_{D_X} D_{X\to Y} \otimes^{\mathbb{L}}_{f^\bullet D_Y} f^\bullet \mathbb{D} M[-\dim Y]) \\
&\cong f_\bullet((\Omega_X \otimes^{\mathbb{L}}_{\mathcal{O}_X} N) \otimes^{\mathbb{L}}_{D_X} D_{X\to Y}) \otimes^{\mathbb{L}}_{D_Y} \mathbb{D} M[-\dim Y] \\
&\cong f_*(\Omega_X \otimes^{\mathbb{L}}_{\mathcal{O}_X} N) \otimes^{\mathbb{L}}_{D_Y} \mathbb{D} M[-\dim Y] \\
&\cong (\operatorname{RHom}_{D_Y}(M, D_Y) \otimes^{\mathbb{L}}_{\mathcal{O}_Y} \Omega_Y^{-1}) \otimes^{\mathbb{L}}_{D_Y} f_*(\Omega_X \otimes^{\mathbb{L}}_{\mathcal{O}_X} N) \\
&\cong \operatorname{RHom}_{D_Y}(M, D_Y) \otimes^{\mathbb{L}}_{D_Y} (f_*(\Omega_X \otimes^{\mathbb{L}}_{\mathcal{O}_X} N) \otimes^{\mathbb{L}}_{\mathcal{O}_Y} \Omega_Y^{-1}) \\
&\cong \operatorname{RHom}_{D_Y}(M, f_*(\Omega_X \otimes^{\mathbb{L}}_{\mathcal{O}_X} N) \otimes^{\mathbb{L}}_{\mathcal{O}_Y} \Omega_Y^{-1}) \\
&\cong \operatorname{RHom}_{D_Y}(M, f_* N),
\end{aligned}$$

where we used [HTT08, Proposition 2.6.14] for the first line, [HTT08, Proposition 1.5.19] for the third line, the projection formula [HTT08, Proposition C.2.6] for the fifth line, [HTT08, Proposition 1.5.19] again for the eighth line, [HTT08, Proposition 2.6.13] for the ninth line and the commutative square for pushforward and side-changing [HTT08, Page 23] for the last line.

□

- All functors preserve holonomicity, [HTT08, Section 3.2].
- The functor $\mathbb{D}$ is exact on *holonomic* D-modules, [HTT08, Proposition 3.2.1].
- Restricting to *holonomic* D-modules, we have

$$f_! \dashv f^!,$$

and

$$f^* \dashv f_*,$$

regardless of the properness and smoothness of f, [HTT08, Theorem 3.2.14].

Chapter 2

On the derived ring of differential operators on a singularity

2.1 D-modules as modules over a DG algebra

Although when X is singular the category $D\text{-mod}_X$ can no longer be viewed as the category of modules over $\mathrm{Diff}(X)$, in this section, we introduce a better substitute for $\mathrm{Diff}(X)$. This substitute will in general be a DG algebra rather than a usual ring.

In [BN04], they showed in the case when X has only cuspidal singularities one can still use the ring of differential operators $\mathrm{Diff}(X)$ and the abelian category $D\text{-mod}_X$ is equivalent to $\mathrm{Diff}(X)$-mod. We show that our DG algebra reduces to $\mathrm{Diff}(X)$ and our equivalence reduces to the derived version of theirs in this case.

This section is divided into three parts: the first part deals with the general case, the second part deals with the cuspidal case and the last part deals with a vanishing result that we will need for Section 2.4.

2.1.1 General case

Suppose X is affine. We can choose $i : X \hookrightarrow V$ a closed embedding into a smooth affine variety V (most of the time $V = \mathbb{A}^n$); note that if X is smooth, we can just take V to be X. Recall that we have defined the Kashiwara's category $D\text{-mod}_X$ to be the full subcategory of D-modules on V that are set-theoretically supported on X. It can be shown that this definition does not depend on the embedding i, that is if $i_k : X \hookrightarrow V$ are closed embeddings into smooth affine variety V_k for $k = 1, 2$, then the full subcategory of D-modules on V_1 that are set-theoretically supported on X is equivalent to the full subcategory of D-modules on V_2 that are set-theoretically supported on X. See [ES17, Corollary A.9]. We define the following element of

D-mod$_X$:

$$D_X := ID_V \backslash D_V,$$

where I is the defining ideal of X. This is clearly a right D_V-module that is supported on X, hence by Kashiwara's definition, an element of D-mod$_X$. The module D_X has the defining property $\mathrm{Hom}(D_X, M) = \Gamma_X(M)$, the vector space of sections of M scheme-theoretically supported on X (*i.e.*, annihilated by I for some n). Note that if X and V are smooth, D_X is just the usual transfer module $D_{X \to V}$. The object D_X does not depend on the choice of embedding. Indeed, given two closed embeddings $i_k : X \hookrightarrow V_k$ for $k = 1, 2$, let I_{kX} be the ideal defining X in V_k and $D_{X,k} := I_{kX} D_{V_k} \backslash D_{V_k}$. One can check that the equivalence of categories in [ES17, Theorem A.8] sends $D_{X,1}$ to $D_{X,2}$. When X is not affine, we glue the categories of the open subsets U_i of a covering together to obtain a canonical abelian category of D-modules on X. The local objects D_{U_i} glue together in a canonical way to get a global D-module. See [ES17, Section A.2] and [Bra+, Section 1.7.2].

We recall that an object E in the derived category $\mathcal{T}$ of an abelian category $\mathcal{A}$ is called a *generator* if $\mathrm{Hom}(E[i], M) = 0$ for all $i \in \mathbb{Z}$, implies $M = 0$. The category $\mathcal{T}$ is called *cocomplete* if it has arbitrary direct sums. An object $C \in \mathcal{T}$ is called *compact* if $\mathrm{Hom}(C, -)$ commutes with direct sums. See [Lun10, Section 2.1].

We fix X with a closed embedding into V. Recall that D_V-mod$_X$ is the abelian category of quasi-coherent D-modules on V supported on X and let $D_{D\text{-mod}_X}(D_V)$ be the full subcategory of $D(D_V)$ consisting of complexes with cohomology sheaves supported on X.

The theorem below is a special case of a result of Gaitsgory–Rozenblyum, as we will explain, but with a more explicit proof.

Theorem 2.1.1. Let X be an affine variety, then the module D_X is a compact generator in $D(D_V\text{-mod}_X)$.Ask Travis if this is ok.

Proof. Generation: To show it is a generator, it is enough to observe that D_X has a nonzero map to every nonzero D-module M supported on X, as then there is a map from $D_X[-i]$ a nonzero complex with a nonzero term M sitting in degree i. Take M to be a non-zero D-module supported on X, then because every element is annihilated by I^n for some n, for $0 \neq m \in M$, we can choose n to be such that $I^n \cdot m = 0$ and $I^{n-1} \cdot m \neq 0$, choose $m' \in I^{n-1} \cdot m$. Hence there is a non-zero map sending $1 \in D_X$ to this element m'.

Compactness: To show it is compact in $D(D_V\text{-mod}_X)$, it is enough to show it is compact in $D(D_V)$ as $D(D_V\text{-mod}_X)$ is a full subcategory. Recall compactness is equivalent to perfectness in derived categories of rings [SP, Proposition 15.78.3] and a perfect complex is a finite complex of locally projective objects. Since $\mathcal{O}_V$ has finite global dimension, we can take a finite projective resolution $P^\bullet$ of $\mathcal{O}_X$ as an $\mathcal{O}_V$-mod. Consider $P^\bullet \otimes_{\mathcal{O}_V} D_V$. This complex is an object in $D_{\mathcal{A}}(D_V)$ because $\operatorname{supp}(M \otimes_{\mathcal{O}_V} N) = \operatorname{supp} M \cap \operatorname{supp} N$. Since $D_X = \mathcal{O}_X \otimes_{\mathcal{O}_V} D_V$, we have that $P^\bullet \otimes_{\mathcal{O}_V} D_V$ is a finite projective D-module resolution of D_X as D_V is flat over $\mathcal{O}_V$. This completes the proof. □

Remark 2.1.2. If X is not affine, this construction still produces a compact object which is locally a generator.

Remark 2.1.3.
1. In [GR14, Corollary 3.3.3], the authors proved a more general statement (but they require more difficult preliminaries) than Theorem 2.1.1, for a general variety X (not necessarily affine), replacing $\mathcal{O}_X$ by a compact generator M of $\mathcal{O}_X$-mod, so that the compact generator of D-mod is the *induction* of M. This induction makes sense in general, but in the case that X is embedded into a smooth affine variety V, it is $i_*M \otimes^{\mathbb{L}}_{\mathcal{O}_V} D_V$.

2. Furthermore, in [GR14, Proposition 4.7.3] they proved that for X is a variety, with a closed embedding into V. Then the inclusion functor

$$i : D(D_V\text{-mod}_X) \to D_{D\text{-mod}_X}(D_V) \tag{†}$$

is an equivalence of categories. In particular, it is fully faithful.

Note that in [GR14] it is stated that $i : D(\mathrm{Crys}^r(X)^\heartsuit) \to \mathrm{Crys}^r(X)$ is an equivalence, where $\heartsuit$ denotes the heart of the t-structure. This is equivalent to our statement because by [GR14, Section 5.5] the category of right crystals $\mathrm{Crys}^r(X)$ can be canonically identified with the (derived) category of right D-modules on X, and furthermore by Kashiwara's Lemma [GR14, Proposition 2.5.6] $\mathrm{Crys}^r(X)$ can be identified with crystals on V supported on X.

This result can be thought of as an analogue of Beilinson's result for perverse sheaves [Bei87, Theorem 1.3]: the derived category of the abelian category of perverse sheaves is the derived constructible category.

This theorem is important because it shows that two natural derived categories of D-modules are equivalent.

An abelian category $\mathcal{C}$ is called a *Grothendieck category* if it has a g-object, small colimits and the filtered colimits are exact. Recall that an object $G \in \mathcal{C}$ is called a *g-object* if the functor $C \to \operatorname{Hom}_{\mathcal{C}}(G, C)$ is conservative, *i.e.* $C_1 \to C_2$ is an isomorphism as soon as $\operatorname{Hom}(G, C_1) \to \operatorname{Hom}(G, C_2)$ is an isomorphism. In the case of a cocomplete abelian category, this is equivalent to saying that every object C of $\mathcal{C}$ admits an epimorphism $G^{(S)} \to C$, where $G^{(S)}$ denotes a direct sum of copies of G, one for each element of the (possibly infinite) set S. Such an object G is usually called a generator, but we already used this term previously. For more detail, see [Lun10, Section 2.4].

Note that the abelian category of quasi-coherent D-modules (*i.e.* quasi-coherent after forgetting to $\mathcal{O}$-modules) on an affine variety X is a Grothendieck category. This fact is mentioned in [GR14, Section 4.7], but we give more detail here. We only need to show it has a g-object, as the other axioms are obvious. We let $G = \bigoplus_n I^n D_V \backslash D_V$, where $i : X \to V = \mathbb{A}^n$ is a closed embedding and I is the defining ideal. This is a g-object since if M is supported on X, then every element is killed by some element in I^n. It implies that there is a surjective map from $G^{(S)}$ to M.

Remark 2.1.4. As any Grothendieck category has enough injectives, the above implies that the abelian category of D-modules on a variety X has enough injectives.

We recall the following fact about Grothendieck categories (see [Kel94, Lemma 4.2, Theorem 4.3], but we are using the version found in [Lun10, Proposition 2.6]):

Proposition 2.1.5. Let $\mathcal{A}$ be a Grothendieck category such that the triangulated category $D(\mathcal{A})$ has a compact generator E. Then the functor $\mathrm{RHom}(E, -) : D(\mathcal{A}) \to D(\mathrm{REnd}(E)\text{-mod})$ is an equivalence of categories.

Here $D(\mathrm{REnd}(E)\text{-mod})$ denotes the derived category of right DG modules, which is the localisation of homotopy category $Ho(\mathrm{REnd}(E))$ with respect to quasi-isomorphisms, see [Lun10, Section 2.3].

Remark 2.1.6. As the inverse functor of an equivalence is always given by the adjoint, the inverse functor to $\mathrm{RHom}(E, -)$ is given by $M \mapsto M \otimes_{\mathrm{REnd}(E)} E$.

Notice that in the case $H^\bullet(\mathrm{REnd}(E))$ is bounded in degree, the functor $\mathrm{RHom}(E, -)$ maps $D^b(\mathcal{A})$ into $D^b(\mathrm{REnd}(E)\text{-mod})$. This is also an equivalence since the inverse also maps $D^b(\mathrm{REnd}(E)\text{-mod})$ into $D^b(\mathcal{A})$.

Combining Theorem Theorem 2.1.1 and Proposition 2.1.5, we get the following corollary.

Corollary 2.1.7. There is an equivalence of categories between the bounded derived category of quasi-coherent D-modules on an affine variety X and the bounded derived category of DG modules over $\mathrm{REnd}(D_X)$:

$$D^b(D\text{-mod}_X) \cong D^b(\mathrm{REnd}(D_X)\text{-mod}).$$

This gives a proof of Theorem 1.1.1: define $\mathrm{Diff}(X)^{dg} := \mathrm{REnd}(D_X)$. We get a triangulated equivalence

$$D(D\text{-mod}_X) \cong D(\mathrm{Diff}(X)^{dg}\text{-mod}).$$

Remark 2.1.8. If X is not affine, but $X \hookrightarrow V$ is still a closed embedding into a smooth variety, as we explained at the beginning of the section that D_X can still be defined. In this case, we still have an equivalence of $D(\mathcal{A}) \to \mathcal{REnd}(D_X)$-mod, where $\mathcal{REnd}(D_X)$-mod is a category of sheaves of modules on X.

Recall Grothendieck's filtration on $\mathrm{Diff}(X)$ for any variety X:

$$\mathrm{Diff}(X) = \bigcup_{n \geq 0} \mathrm{Diff}(X)_n,$$

where

1. $\mathrm{Diff}(X)_0 := \mathcal{O}(X)$,
2. $\mathrm{Diff}(X)_n := \{d \in \mathrm{End}_{\mathbb{C}}(\mathcal{O}(X)) | [d, \mathcal{O}(X)] \subset \mathrm{Diff}(X)_{n-1}\}$.

Note that D_X is also filtered, as it is a quotient of filtered modules. Therefore $\mathrm{End}(D_X)$ is also filtered, where the filtration is given by

$$\mathrm{End}(D_X)_n := \{f \in \mathrm{End}(D_X) | f((D_X)_k) \subset (D_X)_{k+n}\},$$

where $(D_X)_k$ denotes the kth filtered piece of D_X.

The following description of the (underived) endomorphisms of D_X, proved by an explicit computation on V, has been known to experts for a long time (see [MRS01, Theorem 15.3.15], [Bra+, Theorem 1.7.1]). While the formula is old, we got the idea to think of it in terms of the object D_X from [Bra+].

Theorem 2.1.9. There is a canonical filtered isomorphism

$$\phi : \mathrm{End}(D_X) \to \mathrm{Diff}(X).$$

As the algebra $\mathrm{End}(D_X)$ sits in our DG algebra $\mathrm{REnd}(D_X)$ in degree zero and by the Theorem it is isomorphic to the usual ring of differential operators, we see that one to to view the the higher DG structure of $\mathrm{REnd}(D_X)$ is that it detects singularities and serves as a correction to $\mathrm{Diff}(X)$ in the singular case.

2.1.2 Cuspidal Case

We now turn the attention to the cuspidal case. Recall we say $f : Y \to X$ is a *universal homeomorphism* if for every morphism $Y' \to Y$ the pullback $f_{(Y')} : Y' \times_Y X \to Y'$ is a homeomorphism. Equivalently, $f : X \to Y$ of k-varieties is a universal homeomorphism if and only if f satisfies:

1. f is a finite morphism.

2. f is surjective.

3. For every algebraically closed field K, the map $X(K) \xrightarrow{f(K)} Y(K)$ is injective.

Definition 2.1.10. We say $f : Y \to X$ is a *cuspidal quotient morphism* if it is a universal homeomorphism and X and Y are Cohen–Macaulay. It is a *good* cuspidal quotient morphism if, in addition, a certain local cohomology sheaf vanishes, which is automatically satisfied if X (or Y) is a smooth variety. We say a Cohen–Macaulay variety X is cuspidal if there is a cuspidal quotient morphism from a smooth variety to X. In the curve case, this is equivalent to the normalisation map being a bijective resolution of singularities. Note the definition of cuspidal includes the case of smooth varieties. See [BN04, Section 2] and the references therein.

Examples of cuspidal quotient morphisms include the normalization map of a curve with cusp singularities, the normalization map $\mathfrak{h} \to X_m$ of the space of quasiinvariants for a Coxeter group, and the Frobenius homeomorphism in characteristic p, see [BN04, Section 1.2]. There are also examples from the geometry of Lie algebras, see [Los22, Theorem 4.4].

In [BN04], the authors used Diff(X)-modules rather than Kashiwara's category D-mod$_X$. Let $\mathrm{Diff}_X(M, N)$ be differential operators from M to N, where M and N are $\mathcal{O}_X$-modules. Suppose $f : Y \to X$ be a general morphism, the authors defined transfer bimodules $D^{BN}_{X\leftarrow Y}$ and $D^{BN}_{Y\to X}$ as duals of *jets* [BN04, Definition 2.5, 2.12].

Their key results in the affine case are summarised in the following theorem:

Theorem 2.1.11. If $f : Y \to X$ is a good cuspidal quotient morphism, then the following hold:

(1) If Y is smooth, and consider a closed embedding $i : X \to Z$ into a smooth Z, then the D-module pushforward f_* and pullback $f^!$ induces an equivalence:

$$D\text{-mod}_Y \cong D\text{-mod}_X,$$

[BN04, Proposition 3.24].

(2) The bimodules transfer bimodules $D^{BN}_{X\leftarrow Y}$, $D^{BN}_{X\to Y}$ induce Morita equivalences

of the categories of (left or right) Diff(Y)-modules and Diff(X)-modules:

$$\text{Diff}(Y)\text{-mod} \cong \text{Diff}(X)\text{-mod}.$$

Furthermore, $D^{BN}_{X \leftarrow Y}$ is projective as a left module over Diff(X) and as a right module over Diff(Y) [BN04, Theorem 4.3].

(3) If Y is smooth, $D\text{-mod}_X \cong \text{Diff}(X)\text{-mod}$ [BN04, Corrolary 4.4].

(4) $D^{BN}_{X \leftarrow Y} = \text{Diff}_X(\mathcal{O}_Y, \mathcal{O}_X)$ [BN04, Corollary 2.14].

Note that (3) follows from (1) and (2), and we will strengthen (1) in Proposition 2.1.14.

Remark 2.1.12. Since, by our definition, varieties are reduced, in the curve case the Cohen–Macaulay condition is automatically satisfied. The above theorem generalises the curve case result found in [SS88] saying that the category of D-modules on a cuspidal curve is Morita equivalent to the category of D-modules on its (smooth) normalization. This is a generalisation, because for cuspidal curves, the normalisation map is a universal homeomorphism, which is a cuspidal quotient morphism.

Theorem 2.1.13. *If X is cuspidal, then D_X is sent to Diff(X) under the equivalence of 2.1.11(3). Furthermore,*

$$\text{REnd}(D_X) \cong \text{End}(D_X) \cong \text{Diff}(X).$$

Therefore, the functor $\text{REnd}(D_X, -)$ in Proposition 2.1.5 is the derived functor of the abelian equivalence in Theorem 2.1.11(3).

We see that $\text{REnd}(D_X)$ has vanishing higher cohomology when X is not smooth but cuspidal. This shows $\text{REnd}(D_X)$ does not detect all the singularities in the sense that there will be no higher DG structure and $\text{REnd}(D_X) \cong \text{Diff}(X)$ in degree 0 if X is a cuspidal but not smooth.

Proof. Consider $Y \to X \hookrightarrow \mathbb{A}^n$, with both Y and $\mathbb{A}^n$ smooth and $f : Y \to X$ is a cuspidal morphism, $i : X \hookrightarrow \mathbb{A}^n$ is a closed embedding. Let I be the ideal defining X.

Firstly, we show that $f^! D_X = D^{BN}_{X \leftarrow Y}$. Since $f^!$ has an inverse, it is equivalent to show $f_* D^{BN}_{X \leftarrow Y} = D_X$, which by definition means that $i_* f_* D^{BN}_{X \leftarrow Y} = ID_{\mathbb{A}^n} \backslash D_{\mathbb{A}^n}$. This means that $D^{BN}_{X \leftarrow Y}$ is sent to D_X under the equivalence of 2.1.11(1).

We have:

$$
\begin{aligned}
(i \circ f)_* D^{BN}_{X \leftarrow Y} =& D^{BN}_{X \leftarrow Y} \otimes_{D_Y} D_{Y \to \mathbb{A}^n} \\
=& D^{BN}_{X \leftarrow Y} \otimes_{D_Y} \mathcal{O}_Y \otimes_{\mathcal{O}_{\mathbb{A}^n}} D_{\mathbb{A}^n} \\
=& D^{BN}_{X \leftarrow Y} \otimes_{D_Y} \mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathrm{Diff}(X) \otimes_{\mathrm{Diff}(X)} \mathcal{O}_X \otimes_{\mathcal{O}_{\mathbb{A}^n}} D_{\mathbb{A}^n} \\
=& D^{BN}_{X \leftarrow Y} \otimes_{D_Y} D^{BN}_{Y \to X} \otimes_{\mathrm{Diff}(X)} \mathcal{O}_X \otimes_{\mathcal{O}_{\mathbb{A}^n}} D_{\mathbb{A}^n} \\
=& \mathrm{Diff}(X) \otimes_{\mathrm{Diff}(X)} ID_{\mathbb{A}^n} \backslash D_{\mathbb{A}^n} \\
=& ID_{\mathbb{A}^n} \backslash D_{\mathbb{A}^n},
\end{aligned}
$$

where we used [BN04, Lemma 4.2(2)] for $\mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathrm{Diff}(X) = D^{BN}_{Y \to X}$ and [BN04, Theorem 4.3] for $D^{BN}_{X \leftarrow Y} \otimes_{D_Y} D^{BN}_{Y \to X} = \mathrm{Diff}(X)$.

By [BN04, Theorem 4.3] again, $D^{BN}_{X \leftarrow Y}$ is sent to $\mathrm{Diff}(X)$ under the equivalence of 2.1.11(2), this must mean that $\mathrm{Diff}(X)$ and D_X are equivalent under the equivalence of Theorem 2.1.11 (3).

Next, because f induces an equivalence between D-mod_Y and D-mod_X, we must have that for $i > 0$:

$$
\begin{aligned}
\mathrm{Ext}^i_{D\text{-mod}_X}(D_X, M) &= \mathrm{Ext}^i_{D\text{-mod}_Y}(f^! D_X, f^! M) \\
&= \mathrm{Ext}^i_{\mathrm{Diff}(Y)\text{-mod}}(D^{BN}_{X \leftarrow Y}, f^! M) \\
&= 0,
\end{aligned}
$$

where for the last line, we used that $D^{BN}_{X \leftarrow Y}$ is projective in the good cuspidal case. And hence in the good cuspidal case, the functor $M \mapsto \mathrm{RHom}(D_X, M)$ from $D^b(\mathcal{A})$ to $D^b(\mathrm{REnd}(D_X)\text{-mod})$ is actually an abelian functor, *i.e.*, it restricts to an exact functor of abelian categories from D-mod_X to $\mathrm{End}(D_X)$-mod (which is $\mathrm{Diff}(X)$-mod by Theorem 2.1.9). The module D_X is mapped to $\mathrm{Diff}(X)$.

Since both abelian equivalences send the compact projective generator D_X to the same compact projective generator $\mathrm{Diff}(X)$, by embedding D-mod_X into D-mod_V and using Eilenberg–Watts theorem ([Eil60] [Wat60]), the functor $\mathrm{Hom}(D_X, -)$ is the only possible equivalence. Therefore our equivalence reduces to the equivalence in [BN04] in the cuspidal case.

□

The D-module equivalence in Theorem 2.1.11 (1) can be easily generalised to remove the good condition at the cost of getting a derived equivalence rather than an abelian equivalence. We are going to use the following proposition in Section 2.3.

Proposition 2.1.14. Suppose $f : Y \to X$ is a universal homeomorphism, then there is a derived equivalence between D-modules on Y and D-modules on X.

We mimic the proof of [BN04, Proposition 3.14].

Proof. Consider the Cartesian diagram

$$\begin{array}{ccc} Y \times_X Y & \xrightarrow{p_1} & Y \\ \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{f} & X. \end{array}$$

Since f is proper (because f is a universal homeomorphism, it is universally closed and of finite type, and we have assumed separatedness already), we can use proper base change: $p_{1*}p_2{}^! = f^! f_*$, see [Gai13, Proposition 5.4.2]. Note that a D-module on $Y \times_X Y$ is the same as a D-module on $Y \times Y$ set-theoretically supported on the diagonal $\Delta : Y \to Y \times Y$. Indeed $(Y \times_X Y)^{\mathrm{red}} = \Delta(Y)^{\mathrm{red}}$. The maps $p_2^!$ and p_{1*} can be identified with the pullback and pushforward of $\pi : \Delta(Y) \to Y$. Therefore, by embedding Y into a smooth variety we see that by Kashiwara's equivalence, we have $p_{1*}p_2{}^! = \mathrm{Id}$ and hence by proper base change $f^! f_* = \mathrm{Id}$.

To show that $f_* f^! = \mathrm{Id}$, note that we have a natural map $f_* f^! N \to N$, and complete the cone. We have $f_* f^! N \to N \to M$, and thus we get $f^! f_* f^! N \to f^! N \to f^! M$, which is $f^! N \to f^! N \to f^! M$ by the above paragraph. Hence $f^! M = 0$. Note that f is surjective and dominant. Suppose that $M \neq 0$. Let Z be an irreducible component of the (reduced) support of M. By passing to a smooth dense subset, we may assume that Z is smooth. Let z be the generic point of Z. Then M_z is a nonzero vector space over the residue field $\kappa(z)$. By dominance, there is an irreducible component Z' of Y such that for its generic point z', the induced map $\mathcal{O}_{Z,z} \to \mathcal{O}_{Z',z'}$ is an injection (a field extension). Therefore, $f^! M_z$ is given as:

$$\kappa(z') \otimes_{\kappa(z)} f^{-1}(M_z)[\dim X - \dim Y],$$

which is nonzero (and concentrated in degree 0 as $\dim X = \dim Y$ because f is a universal homeomorphism). This is a contradiction. □

Remark 2.1.15. It follows from Theorem 4.3 and Remark 4.5 of [BN04] that in the good cuspidal case, the derived equivalence is in fact abelian.

We now recall *seminormalisation*.

Definition 2.1.16. If X is a variety, then the seminormalisation X^{sn} is the initial object in the category of universal homeomorphisms $Y \to X$.

Seminormalisation always exists and note that by definition a curve is cuspidal if and only if its seminormalisation coincides with its normalisation. See [SP, Section 29.47].

Remark 2.1.17. A variety X is cuspidal if and only if X^{sn} is smooth. Indeed, by definition X^{sn} is smooth implies that X is cuspidal. Conversely, if $Y \to X$ is a universal homeomorphism from a smooth variety Y, then by the universal property, there is a map $X^{sn} \to Y$, and since this is a finite birational map (as it factors through one), as Y is normal (as it is smooth), Zariski's main theorem implies that $X^{sn} \cong Y$ and hence X^{sn} is smooth. Ask Travis if this is OK

We have the following corollary.

Corollary 2.1.18. Let $f : X^{sn} \to X$ be the seminormalisation map. There is a derived equivalence

$$D^b(D\text{-mod}_X) \cong D^b(D\text{-mod}_{X^{sn}}),$$

where the equivalence is induced by f_* and $f^!$.

2.1.3 Vanishing Ext for D-modules

For a general X, we have the following vanishing result:

Proposition 2.1.19. If $M \in D\text{-mod}_X$ is supported at a point then

$$\mathcal{E}\mathrm{xt}^i_{D(D\text{-mod}_X)}(D_X, M) = 0$$

for $i \geq 1$.

Proof. Since by embedding X into a smooth V, $D\text{-mod}_X$ is a subcategory of $\mathrm{Diff}(V)$-mod, we can do the calculation in the later category. If M is supported at a point, then M is just a direct sum of delta modules (i.e., $M \cong i_{pt_*}(\mathbb{C})$ where i_{pt} is the inclusion map of the point). We can restrict to a local calculation in a formal neighbourhood V of the origin with coordinates $x_1, \ldots, x_n$ and assume without loss of generality that the point is the origin and that $M = \mathbb{C}[\partial_{x_1}, \ldots, \partial_{x_n}]$. As it is shown in [Mat87, Application 3], it is the injective hull of $\mathbb{C}$ in $\mathbb{C}[\![x_1, \ldots, x_n]\!]$.

We claim that

$$\mathbb{C}[x_1, \ldots, x_n] \hookrightarrow \mathbb{C}[\![x_1, \ldots, x_n]\!]$$

is a flat ring extension. Indeed $\mathbb{C}[\![x_1, \ldots, x_n]\!]$ is the product of a countable family of copies of $\mathbb{C}[x_1, \ldots, x_n]$ as $\mathbb{C}[x_1, \ldots, x_n]$-modules. It is known that for any ring A, the direct product of any family of flat A-modules is flat if and only if the ring A is *coherent*, that is, every finitely generated ideal is finitely presented. See [Cha60, Theorem 2.1]. In our case, the ring $\mathbb{C}[x_1, \dots, x_n]$ is Noetherian by Hilbert Basis Theorem, hence is also coherent. This proves the claim. Is this necessary?

As we showed M is an injective $\mathbb{C}[\![x_1, \dots, x_n]\!]$-module and $\mathbb{C}[x_1, \dots, x_n] \hookrightarrow \mathbb{C}[\![x_1, \dots, x_n]\!]$ is a flat ring extension, by [SP, Lemma 10.39.4], M is also an injective $\mathbb{C}[x_1, \dots, x_n]$-module.

Then by the derived tensor-hom adjunction [Wei94, Theorem 10.8.7], we have

$$\mathrm{RHom}_{D(D_V)}(\mathcal{O}_X \otimes^{\mathbb{L}}_{\mathcal{O}_V} D_V, M) \cong \mathrm{RHom}_{D(\mathcal{O}_V)}(\mathcal{O}_X, \mathrm{RHom}_{D(D_V)}(D_V, M)).$$

As D_V is flat over $\mathcal{O}(V)$, taking the i-th cohomology we get

$$\mathrm{Ext}^i_{D_V}(D_X, M) \cong \mathrm{Ext}^i_{\mathcal{O}_V}(\mathcal{O}_X, M),$$

which is 0 as M is injective. $\square$

Remark 2.1.20. More generally, the injective dimension of M as an $\mathcal{O}$-module is at most $\dim \mathrm{supp}\, M$ for a general D-module. See [Lyu00, Main Theorem] and [Lyu93, Theorem 2.4]. Therefore we get $\mathrm{Ext}^i(D_X, M) = 0$ for $i > \dim \mathrm{supp}(M)$.

2.2 Calculation of cohomology in the hypersurface case

In this section, we restrict to the case where we have a hypersurface X that is cut out by a single equation f in $\mathbb{A}^n$. We wish to calculate the cohomology of $\mathrm{RHom}(D_X, D_X)$ or more generally $\mathrm{RHom}(D_X, M)$, where M is a D-module supported on X. We will see that $\mathrm{Ext}^i(D_X, M)$ will vanish for $i \geq 2$. In the hypersurface case we can write down the formula for $\mathrm{Ext}^1(D_X, M)$ easily once we have the correct derived category of D-modules supported on X.

There is a free resolution $0 \to D_{\mathbb{A}^n} \to D_{\mathbb{A}^n} \to D_X \to 0$, where the first map is applying multiplication by f on the left and the second map is the quotient map. Note here we are invoking (†) of Remark 2.1.3(2).

We can replace the object D_X with its free resolution $D_{\mathbb{A}^n} \xrightarrow{f\cdot} D_{\mathbb{A}^n}$. By Theorem 2.1.3, $\mathrm{RHom}_{D^b(\mathcal{A})}(D_X, M)$ is isomorphic to

$$\mathrm{RHom}_{D^b_{\mathcal{A}}(D_V)}(D_{\mathbb{A}^n} \xrightarrow{f\cdot} D_{\mathbb{A}^n}, M).$$

Since $D^b_{\mathcal{A}}(D_V)$ is defined as the full subcategory of $D^b(D_V)$, this is

$$\mathrm{RHom}_{D^b(D_V)}(D_{\mathbb{A}^n} \xrightarrow{f\cdot} D_{\mathbb{A}^n}, M).$$

The complex is then $M \to M$ where the map now is applying multiplication by f on the right. Therefore,

$$\mathrm{Ext}^0(D_X, M) = M^f := \{m \in M, mf = 0\},$$

$$\mathrm{Ext}^1(D_X, M) = M/Mf,$$

in particular

$$\mathrm{Ext}^1(D_X, D_X) = \frac{D_{\mathbb{A}^n}}{D_{\mathbb{A}^n} \cdot f + f \cdot D_{\mathbb{A}^n}}.$$

Note this is only a vector space, not a $D_{\mathbb{A}^n}$-module.

Recall by Theorem 2.1.13, $\mathrm{Ext}^1(D_X, M) = 0$ for cuspidal X. This shows for cuspidal singularities, $\frac{D_{\mathbb{A}^n}}{D_{\mathbb{A}^n} \cdot f + f \cdot D_{\mathbb{A}^n}}$ is 0, for which we can't find a purely algebraic proof.

To summarise, we have the following:

Formula 2.2.1. In the case X is defined by a single equation f, we have that

1. $\mathrm{Hom}(D_X, M) = (M)^f$.
2. $\mathrm{Ext}^1(D_X, M) = M/Mf$.
3. $\mathrm{Ext}^{\geq 2}(D_X, M) = 0$.
4. When X is a cuspidal hypersurface (automatically CM and including smooth) then
$$\frac{D_{\mathbb{A}^n}}{D_{\mathbb{A}^n} \cdot f + f \cdot D_{\mathbb{A}^n}} = 0.$$

Even though $\mathrm{Ext}^1(D_X, M)$ is in general not a $D_{\mathbb{A}^n}$-module, it is however still an $\mathrm{REnd}(D_X)$-module, where now we have viewed it as a module over a DG algebra. Taking cohomology we have:

Lemma 2.2.2. $\mathrm{Ext}^1(D_X, M)$ is a module over $\mathrm{Ext}^\bullet(D_X, D_X)$ via the Yoneda product.

An interesting question is when $\mathrm{Ext}^1(D_X, M)$ is zero. In this case, $\mathrm{RHom}(D_X, M) \cong \mathrm{Hom}(D_X, M)$ can be viewed as an ordinary module over $\mathrm{Diff}(X)$ (with possible higher A_∞ structure).

2.2.1 Action of cohomology in the hypersurface case

Lemma 2.2.3. The action of $\mathrm{End}(D_X)$ on $\mathrm{Hom}(D_X, M)$ is usual composition; the action of $\mathrm{End}(D_X)$ on $\mathrm{Ext}^1(D_X, M)$ is a *twisting* action (see the proof below). And the action of $\mathrm{Ext}^1(D_X, D_X)$ on $\mathrm{Hom}(D_X, M)$ is the usual multiplication upon identifying $\mathrm{Ext}^1(D_X, D_X)$ with $D_X/D_X f$.

Proof. Note that

$$\begin{aligned}\mathrm{REnd}(D_X) =& \mathrm{RHom}(D_{\mathbb{A}^n} \to D_{\mathbb{A}^n}, D_{\mathbb{A}^n} \to D_{\mathbb{A}^n})\\ =& D_{\mathbb{A}^n} \to D_{\mathbb{A}^n} \oplus D_{\mathbb{A}^n} \to D_{\mathbb{A}^n},\end{aligned}$$

where the first arrow and second arrow are given explicitly by

$$a \mapsto fa \oplus af, \alpha \oplus \beta \mapsto -f\beta + \alpha f. \tag{2.1}$$

Note there is a quasi-isomorphism

$$\begin{array}{ccccc} D_{\mathbb{A}^n} & \longrightarrow & D_{\mathbb{A}^n} \oplus D_{\mathbb{A}^n} & \longrightarrow & D_{\mathbb{A}^n} \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & D_X \oplus 0 & \xrightarrow{-\cdot f} & D_X, \end{array}$$

where the vertical maps are either the zero map ($D_{\mathbb{A}^n} \to 0$) or the projection map ($D_{\mathbb{A}^n} \to D_X$).

Multiplication is given by

$$(a_1, b_1 \oplus c_1, d_1)(a_2, b_2 \oplus c_2, d_2) = (a_1c_2 + b_1a_2, a_1d_2 + b_1b_2 \oplus c_1c_2 + d_1a_2, c_1d_2 + d_1b_2)$$

and its action on $\mathrm{RHom}(D_X, M) = M \to M$ is given by

$$(e, g) \cdot (a, b \oplus c, d) = (ga + eb, gc + ed) \tag{2.2}$$

Now we can calculate the action of $\mathrm{End}(D_X) = (D_X)^f$ on $\mathrm{Hom}(D_X, M) = M^f$ and $\mathrm{Ext}^1(D_X, M) = M/Mf$, via the action and the quasi-isomorphism described above.

From the map (2.1), we get the expression

$$\mathrm{End}(D_X) = \frac{\{(\alpha, \beta) | \alpha f = f\beta\}}{\{f \cdot a \oplus a \cdot f | a \in D_{\mathbb{A}^n}\}}.$$

The algebra $\mathrm{End}(D_X)$ is a quotient of a subspace of $D_{\mathbb{A}^n} \oplus D_{\mathbb{A}^n}$ and $\mathrm{Hom}(D_X, M)$ is a subspace of M, therefore the action is induced by equation (2.2): $e \cdot (b \oplus c) = eb$. Note the isomorphism

$$\frac{\{(\alpha, \beta) | \alpha f = f\beta\}}{\{f \cdot a \oplus a \cdot f | a \in D_{\mathbb{A}^n}\}} \to (D_X)^f$$

is given by projecting to the first coordinate. The inverse isomorphism is given by first lifting to an element α in $D_{\mathbb{A}^n}$ then solving the equation

$$\alpha f = f\beta \tag{2.3}$$

for β, then sent to the quotient. An alternative basis for $D_{\mathbb{A}^n}$ is given by

$$x_1^{i_1}\dots x_n^{i_n}\partial_{x_1}^{j_1}\dots\partial_{x_n}^{j_n},$$

hence there are no f-torsion elements in $D_{\mathbb{A}^n}$ and the solution to the equation is unique. As α is a lift of $(D_X)^f$ in $D_{\mathbb{A}^n}$, we have $\alpha f = 0$ in D_X, which means $\alpha f = f\beta$ in $D_{\mathbb{A}^n}$ for some β. Hence solutions to equation (2.3) exist. Since the action only depends on the first coordinate, we see that the action of $\mathrm{End}(D_X)$ on $\mathrm{Hom}(D_X, M)$ is the usual multiplication.

On the other hand, the action on $\mathrm{Ext}^1(D_X, M)$ is a bit more complicated, it has a *twist*: $\mathrm{Ext}^1(D_X, M)$ is a quotient of M and the action is induced by $g\cdot(b\oplus c) = gc$. Therefore, the action is given by multiplication by β after solving the equation (2.3). It is worthwhile to remark that β has the same *principal symbol* as α, so after taking the associated graded module (with respect to either the arithmetic[1] or the geometric filtration[2] [Bra+, Page 9]), the action is the same as without taking the twist.

The action of $\mathrm{Ext}^1(D_X, D_X)$ on $\mathrm{Hom}(D_X, M)$ is induced by $e\cdot d = ed$. Upon identifying $\mathrm{Ext}^1(D_X, D_X)$ with D_X/D_Xf we see that the action is just the usual multiplication.

□

Example 2.2.4. We compute $\mathrm{Ext}^1(D_X, D_X)$ explicitly when $X = \mathrm{Spec}\,\mathbb{C}[x,y]/(xy)$, the union of two axes in the plane.

Claim: We claim that an explicit basis is given by

$$P(\partial_x,\partial_y), y\partial_y P(\partial_x,\partial_y),$$

where P is a monomial in ∂_x, ∂_y.

To show this, we will use the Diamond Lemma, see [Sch16, Proposition A.2.5]. Choose any ordering such that $x > y$. Note that $\mathrm{Ext}^1(D_X, D_X) = D_{\mathbb{A}^2}/xyD_{\mathbb{A}^2} + D_{\mathbb{A}^2}xy$.

Proof. The relations are the span of:

[1] $F_0D = k, F_1D = k + \mathrm{span}(x_j, \frac{\partial}{\partial x_j})$, F_iD is the image of $F_1D^{\otimes i}$ under the multiplication map.

[2] $F_0D = k[x_1,\dots,x_n]$, $F_1D = \mathrm{span}(f \in k[x_1,\dots,x_n]; g\frac{\partial}{\partial x_j}$ where $g \in k[x_1,\dots,x_n])$, F_iD is the image of $F_1D^{\otimes i}$ under the multiplication map.

1. xyg, where g is a monomial of $x, y, \partial_x, \partial_y$ (from $xyD_{\mathbb{A}^n}$);

2. xg, where ∂_x is not a factor of g;

3. yg, where ∂_y is not a factor of g;

4. and $(bx\partial_x + ay\partial_y)g + abg$, for all g, with $a = 1 + \deg_{\partial_x} g$ and $b = 1 + \deg_{\partial_y} g$ (from $D_{\mathbb{A}^2}xy$ then subtract from $xyD_{\mathbb{A}^2}$).

clean this up?

Using the last line allows us to get rid of all multiples of $x\partial_x$; in what remains, we can get rid of all multiples of x by the second line.

But there are some redundancies: if we have $(xy\partial_x)g$, we get this to zero by the first line, or to $-(a+1)y((1/b)y\partial_y + 1)g$ by the last line.

This means that we can also get rid of multiples of $y^2\partial_y$. We can get rid of all y's when not a multiple of ∂_y.

These are all the redundancies in applying the reductions

$$xyg \to 0,$$

$$xg \to 0,$$

when g is not a multiple of $\partial_x, yg \to 0$ when g is not a multiple of ∂_y, and

$$x\partial_x g \to -a((1/b)y\partial_y + 1)g.$$

So by the Diamond Lemma, we have found a basis consisting of the remaining expressions, which are of the form $P(\partial_x, \partial_y), y\partial_y P(\partial_x, \partial_y)$. This proves the claim. □

From this basis we see that if we look at the dimensions of the filtered pieces with respect to the additive filtration, the sequence of dimensions is a sum of shifted triangular numbers with one in deg 0, one in deg 2, namely numbers of the form

$$\frac{i(i+1)}{2} + \frac{(i+2)(i+3)}{2}.$$

The first several terms of the sequence is 1,3,7,13,21,31...

We also compute the action of $\mathrm{End}(D_X)$ on $\mathrm{Ext}^1(D_X, D_X)$. Note that elements in $\mathrm{End}(D_X)$ are $h \in D_X$ such that $hxy = 0$ *i.e.* $hxy = xyg$ for some $g \in D_{\mathbb{A}^2}$. Also note that

$$x^i y^j \partial_x^n \partial_y^m xy = x^i y^j (xy\partial_x\partial_y + by\partial_y + mx\partial_x + nm)\partial_x^{n-1}\partial_y^{m-1}.$$

So h has representatives

1. $x^i y^j \partial_x^n \partial_y^m$, where $i, j \geq 1$

2. $y^j \partial_y^m$, where $j \geq 1$, and

3. $y^i \partial_y^n$, where $i \geq 1$,

in $D_{\mathbb{A}^2}$. And as we noted before, the action is given by multiplication of the corresponding $x^{i-1}y^{j-1}(xy\partial_x\partial_y + ny\partial_y + mx\partial_x + nm)\partial_x^{n-1}\partial_y^{m-1}$ with the elements of $\mathrm{Ext}^1(D_X, D_X)$. Note that the action is completely determined by the action of $x\partial_x^n$ and $y\partial_y^m$ since they generate $\mathrm{End}(D_X)$. The corresponding elements are $x\partial_x^n + n\partial_x^{n-1}$ and $y\partial_y^m + m\partial_y^{m-1}$. So the action is

$$x\partial_x^n \cdot \partial_x^i \partial_y^j = x\partial_x^{n+i}\partial_y^j + n\partial_x^{n-1+i}\partial_y^j = n\partial_x^{n-1+i}\partial_y^j$$

$$y\partial_y^m \cdot \partial_x^i \partial_y^j = y\partial_x^i\partial_y^{j+m} + m\partial_x^i\partial_y^{m-1+j}$$

$$x\partial_x^n \cdot y\partial_y\partial_x^i\partial_y^j = xy\partial_x^{n+i}\partial_y^{j+1} + ny\partial_x^{n-1+i}\partial_y^{j+1} = ny\partial_x^{n-1+i}\partial_y^{j+1}$$

$$y\partial_y^m \cdot y\partial_y\partial_x^i\partial_y^j = y^2\partial_x^i\partial_y^{j+m+1} + my\partial_x^i\partial_y^{m-2+j}.$$

Note that this example explicitly shows that $\mathrm{Ext}^1(D_X, M)$ does not vanish in general.

2.3 Holonomic D-modules on curves

In this section, we calculate $\mathcal{E}\mathrm{xt}^1(D_X, M)$ for M a (regular) holonomic module on a curve X. In the general case, we show that if M is simple and has nontrivial monodromies in the normalisations of the preimages of the non-cuspidal singularities then $\mathcal{E}\mathrm{xt}^1(D_X, M)$ vanishes, and we conjecture the converse direction is also true for simple M. We prove the conjecture in the case of X is a planar multicross singularity. We show that for every curve X, there exists a curve with planar multicross singularities X^{pl} such that the derived categories of D-modules on them are equivalent.

We need use the following key lemma for this and the next section:

Lemma 2.3.1. Let X be a variety consists of only isolated singularities. For $i \geq 1$, the sheaf $\mathcal{E}\mathrm{xt}^i_{D_V}(D_X, M)$ is a direct sum of sheaves concentrated on the non-cuspidal singular locus.

Proof. We know that $\mathcal{E}\mathrm{xt}^1(D_X, M) = 0$ if and only if $\mathcal{E}\mathrm{xt}^1(D_X, M)_p = 0$ for all points $p \in X$ by [Har77, Proposition 1.1]. As D_X is a coherent D-module, by [HTT08, Lemma 2.6.4.], for any affine open subset U of X we know that

$$\mathcal{RH}\mathrm{om}_{D_V}(D_X, M)|_U \cong \mathcal{RH}\mathrm{om}_{D_U}(D_X|_U, M|_U).$$

Our question is local, we may assume X is affine. As D_X is finitely generated, by mimicking the proof of [Har77, Proposition 6.8] for $\mathcal{O}$-modules, we see that

$$\mathcal{E}\mathrm{xt}^i_{D_V}(D_X, M)_p \cong \mathrm{Ext}^i_{D_{Vp}}(D_{Xp}, M_p).$$

what is the order of things?

We can calculate the stalk of $\mathrm{Ext}^i_{D_{Vp}}(D_{Xp}, M_p)$ in formal neighbourhoods: by flat base change for Ext [SP, Lemma 10.73.1],

$$\mathrm{Ext}^i_{\hat{D_{Vp}}}(\hat{D_{Xp}}, \hat{M_p}) \cong \mathrm{Ext}^i_{D_{Vp}}(D_{Xp}, \hat{M_p}) \cong \mathrm{Ext}^i_{D_{Vp}}(D_{Xp}, M_p) \otimes_{D_{Vp}} \hat{D_{Vp}},$$

where the hat denotes completion and the last equality follows as $\hat{D_{Vp}}$ is flat over D_{Vp}.

Hence by analysing formally locally, if the point p is a cuspidal point (which include the smooth case), then by Theorem 2.1.11, we have concluded that the higher Ext groups vanish $\mathcal{E}\mathrm{xt}^i(D_X, M)_p = 0$. Therefore, the sheaf is concentrated at the non-cuspidal singular points. The singular points are isolated, so we see we only need to do the calculation locally around each non-cuspidal singular point, and $\mathcal{E}\mathrm{xt}^i(D_X, M)$ must be a direct sum of skyscraper sheaves. □

2.3.1 General curve

Let $j : U \to X$ is an open embedding with U smooth. Recall Corollary 2.1.18 about the equivalence between the categories of D-modules on X and its seminormalisation X^{sn}.

Recall that the intermediate extension $\mathbf{IC}(N)$ for a holonomic D-module N on U is the image of canonical morphism $j_! N \to j_* N$ from the adjunction, where j is the inclusion of the affine locally closed subset U. Every simple holonomic D-module is of the form $\mathbf{IC}(N)$ for a simple integral connection N on some locally closed subset Y. See [HTT08, Theorem 3.4.2 (ii)].

Definition 2.3.2. Let $j : U \to X$ be an affine open embedding and N a holonomic D-module on U. We call an intermediate extension $\mathbf{IC}(N)$ *coclean* if the canonical morphism $\mathbf{IC}(N) \hookrightarrow j_* N$ is an isomorphism.

Note the original definition of *clean* is if we have $j_! N \cong \mathbf{IC}(N)$. See [Ost05, Section 2.3] and the references therein.

Lemma 2.3.3. Let $j : U \to X$ be an affine open embedding and N a holonomic D-module on U. If $M = \mathbf{IC}(N)$ is coclean (that is, $\mathbf{IC}(N) \cong j_* N$), then

$$\mathrm{Ext}^1(D_X, M) = 0.$$

Proof. We have

$$\mathcal{RH}\mathrm{om}(D_X, j_* N) = \mathcal{RH}\mathrm{om}(D_U, N) = \Gamma_U(N),$$

where for the last equality we used U is affine. In particular, $\mathcal{E}\mathrm{xt}^i(D_X, j_* N) = 0$ for all D-modules N on U and $i \geq 1$. □

Example 2.3.4. An example of a coclean extension from $\mathbb{G}_m$ to $\mathbb{A}^1$ is

$$N = (\partial - \lambda/x) D_{\mathbb{G}_m} \backslash D_{\mathbb{G}_m}$$

for $\lambda \notin \mathbb{Z}$, because $j_* N$ is simple. An example of a non-coclean extension is $\mathbf{IC}(\Omega_{\mathbb{G}_m}) = \Omega_{\mathbb{A}^1}$ because $j_* \Omega_{\mathbb{G}_m} = \mathbb{C}[t, t^{-1}] \Omega_{\mathbb{A}^1}$.

Example 2.3.5. Let X be n distinct straight lines in the plane $\mathbb{A}^2$ passing through the origin. For example, n distinct lines spanned by the matrix

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 & 0 \\ 0 & 1 & 2 & \dots & n-3 & n-2 & 1 \end{pmatrix},$$

every other n distinct straight lines in the plane $\mathbb{A}^2$ passing through the origin is obtained by the span of a matrix is the a suitable linear transformation of this one. Recall that the pushforward functor for singular varieties is inherited from the pushforward functor for the ambient smooth varieties. Let $i \circ j$ be the inclusion map from U to $\mathbb{A}^2$, where j is the open embedding from $U = \mathbb{A}^1 \backslash \{0\} \to \mathbb{A}^1$ and i is the closed embedding from $\mathbb{A}^1 \to \mathbb{A}^2$. So the pushforward is

$$(i \circ j)_* N = i_\bullet (j_* N \otimes_{D_{\mathbb{A}^1}} D_{\mathbb{A}^1 \to \mathbb{A}^2}),$$

which is $j_* N \otimes_{\mathbb{C}} \mathbb{C}[\partial_y]$. Since δ_0 is simple as a $D_{\mathbb{A}^1}$ module, if $j_* N$ is simple, then we will have that $j_* N \otimes_{\mathbb{C}} \mathbb{C}[\partial_y]$ is a simple $D_{\mathbb{A}^1} \otimes D_{\mathbb{A}^1} \cong D_{\mathbb{A}^2}$ module. Then the intermediate extension must be the pushforward. But indeed when $\lambda \in \mathbb{Z}$, $j_* N \cong \mathbb{C}[t, t^{-1}] \Omega_{\mathbb{A}^1}$, and for $\lambda \notin \mathbb{Z}$, $j_* N \cong (xd - \lambda) D_{\mathbb{A}^1} \backslash D_{\mathbb{A}^1}$. And the simple submodules are $\mathbb{C}[t] \Omega_{\mathbb{A}^1}$ and $(xd - \lambda) D_{\mathbb{A}^1} \backslash D_{\mathbb{A}^1}$ respectively. We see that $(d - \lambda/x) D_{\mathbb{G}_m} \backslash D_{\mathbb{G}_m}$ is still coclean and $\mathbf{IC}(\Omega_{\mathbb{G}_m})$ is still not.

Definition 2.3.6. We say M has *completely non-trivial monodromy* if for every composition factor L of M which is not supported at a point, the monodromy of L about the preimage under the normalisation map $\nu : \tilde{X} \to X$ of every non-cuspidal singularity of X does not have 1 as an eigenvalue.

We have the following theorem:

Theorem 2.3.7. Let X be a curve and M be a regular holonomic D-module on X with completely non-trivial monodromy. Then $\mathrm{Ext}^i(D_X, M) = 0$ for $i \geq 1$.

Proof. By Lemma 2.3.1, we only need to compute in formal neighbourhoods around non-cuspidal singularities.

If the higher Ext groups between D_X and all composition factors vanish, then by the long exact sequence, $\mathrm{Ext}^i(D_X, M)$ vanishes for $i \geq 1$. Therefore we consider L a simple composition factor of M. Then we can take $L \cong \mathbf{IC}(N)$, where N is an integrable connection on a locally closed subset Y of an irreducible component of $\operatorname{supp} L$ (see the proof of [HTT08, Theorem 3.4.2 (ii)]).

If Y is a point, then L is isomorphic to a delta module, therefore we know the Ext groups $\mathrm{Ext}^i(D_X, L) = 0$ for $i \geq 1$ by Proposition 2.1.19.

Let $\nu : \tilde{X} \to X$ be the normalisation map. By Theorem 2.1.11, we know that cuspidal quotient morphisms preserve indecomposable objects as well as isomorphism classes of objects. So the property of having a coclean extension is preserved under cuspidal quotient morphisms. Since we can detect having a coclean extension locally, we see that a simple regular holonomic D-module has a coclean extension if and only if it has a coclean extension after pulling back to the seminormalisation. But then by Example 2.3.5, we see the only monodromies that has the coclean extension property are the non-trivial ones. This is equivalent to saying that, in the normalisation, all monodromies around all preimages of singularities are nontrivial. □

Hence by Proposition 2.1.5 and Corollary 2.1.9, we see that:

Corollary 2.3.8. The abelian subcategory of regular holonomic D-modules with completely non-trivial monodromy over a curve maps to ordinary modules over $\mathrm{Diff}(X)$ (possibly with A_∞ structure). That is, the image of this abelian subcategory has no higher cohomology under the equivalence of Proposition 2.1.5 and hence can be identified with ordinary modules over $\mathrm{Diff}(X)$ by Corollary 2.1.9.

These are not the only D-modules mapping to ordinary modules over $\mathrm{Diff}(X)$, e.g., $j_*\Omega$ maps to ordinary modules over $\mathrm{Diff}(X)$, but the latter doesn't live in the abelian subcategory, as $\mathrm{Ker}(j_*M \to \delta^n) = \mathbf{IC}(X)$ does not map to an ordinary D-module.

Remark 2.3.9. We also expect the converse of the previous theorem to be true for M simple, *i.e.*, $\mathrm{Ext}^{\geq 1}(D_X, M) = 0$ with M regular holonomic implies that M has completely non-trivial monodromy. However, if $f : W \to X$ is the map from W = n-lines intersection at the origin to X, since the $f^! D_X$ is not D_W, the local calculation doesn't go through. The module $f^! D_X$ should look like $D_{X \leftarrow W}$, but since X is not cuspidal, we cant apply the machinery from [BN04]. For a general cuspidal quotient morphism $f : X \to Y$, the vanishing of $\mathrm{Ext}^{>0}(D_X, f^! M)$ does not appear to imply $\mathrm{Ext}^{>0}(D_Y, M)$ vanishing. (However, as discussed above, if X is smooth, or more generally Y is cuspidal, then $\mathrm{Ext}^{>0}(D_Y, M) = 0$ for all D-modules M on Y.) Note that $f^! D_X$ is not projective because if it is then $\mathrm{Ext}^1(D_X, M) = 0$ for X n-straight line intersecting at the origin and M the trivial module.

2.3.2 Planar multicross singularity case

We take a formal neighbourhood $U^\wedge$ at each singular point of a curve X, and we take its normalisation map $\widetilde{U^\wedge} \twoheadrightarrow U^\wedge$. Since $\widetilde{U^\wedge}$ is a normal 1 dimensional, therefore it is smooth. But a regular local complete 1 dimensional ring must be isomorphic to $\mathbb{C}[\![x]\!]$. So, each connected component of $\widetilde{U^\wedge}$ must be of this form. The normalisation map will factor through the bijective map $Z := \operatorname{Spec} \mathbb{C}[\![x]\!] \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \mathbb{C}[\![x]\!] \to U^\wedge$, where $\mathbb{C}[\![x]\!] \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \mathbb{C}[\![x]\!]$ is the fibered coproduct and it is the coordinate ring of the n axes in $\mathbb{A}^n$. This map is a universal homeomorphism and the source is the seminormalisation $U^{\wedge sn}$. Therefore, a seminormal curve is precisely one whose singularities formally locally look like coordinate axes in an affine space. This is known as *multicross singularity* in the literature, see [LV81].

There is another map from the n axes in $\mathbb{A}^n$ (which we named Z) to n distinct (straight) lines in the plane $\mathbb{A}^2$ (which we are going to call W), c.f. Example 2.3.5.

Thus by Proposition 2.1.14, we have:

Lemma 2.3.10. There is a derived equivalence between D-modules on Z and D-modules on W.

Remark 2.3.11. We stress that these equivalences are not isomorphisms of (DG) rings.

Therefore, up to derived equivalence, we can understand the Ext-algebra locally by the next proposition:

Proposition 2.3.12. Let X be a curve such that X is formally locally equivalent to n-lines intersecting in a plane for each non-cuspidal point. If M is a simple regular holonomic D-module on X, then $\mathrm{Ext}^1(D_X, M) = 0$ if and only if M has non-trivial monodromy about each non-cuspidal singularity or M is supported at a point.

Proof. Recall that the support of a sheaf is a closed subset, and that M is simple implies the support M is irreducible. The irreducible closed subsets of n-lines intersecting at the origin are either a point or a copy of $\mathbb{A}^1$. Therefore, $\mathbf{IC}(\Omega_{\mathbb{G}_m})$, $\mathbf{IC}((d-\lambda/x)D_{\mathbb{G}_m}\backslash D_{\mathbb{G}_m})$ and delta modules are the only possible simple regular holonomic modules on $\mathbb{A}^1$. By Proposition 2.1.19 we know the Ext group vanishes for delta modules.

We can explicitly write down a concrete basis of $\mathrm{Ext}^1(D_W, M)$ in the case of M equals $\mathbf{IC}(\Omega_{\mathbb{G}_m})$ or $\mathbf{IC}((d-\lambda/x)D_{\mathbb{A}^1}\backslash D_{\mathbb{A}^1})$, for $\lambda \notin \mathbb{Z}$, when W is n-lines intersecting at the origin, c.f. Example 2.3.5. Recall by Formula 2.2.1, we have $\mathrm{Ext}^1(D_W, M) = M/Mf$, where f is the defining equation of W in $\mathbb{A}^2$.

For $N = \Omega_{\mathbb{G}_m}$, we see $M = \mathbf{IC}(N) = \mathbb{C}[x] \otimes_{\mathbb{C}} \mathbb{C}[\partial_y]$, viewed as a right module. The multiplication on the right is done component-wise, with the standard action on the x-part and $\partial_y^p \cdot y = p\partial_y^{p-1}$. So, applying multiplication by x increases the exponent in the first coordinate and applying multiplication by y decreases the exponent in the second coordinate. Say $f = y(x+\alpha_1 y)\cdots(x+\alpha_{n-1}y)$, $\alpha_i \in \mathbb{C}$ distinct. To calculate $\mathrm{Ext}^1(D_X, M)$, we need to calculate Mf, the image. Clearly applying multiplication by y is a surjection, and applying multiplication by $x+\alpha_i y$ is not a surjection and the image misses a copy of $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}[\partial_y]$ in $\mathbf{IC}(N) = \mathbb{C}[x] \otimes_{\mathbb{C}} \mathbb{C}[\partial_y]$. Thus, we see that $\mathrm{Ext}^1(D_X, M)$ has a basis $\langle 1, x, \ldots, x^{n-2}\rangle \otimes_{\mathbb{C}} \mathbb{C}[\partial_y]$, which is non-zero if $n > 1$ (*i.e.* it is not cuspidal).

For $N = (d-\lambda/x)D_{\mathbb{G}_m}\backslash D_{\mathbb{G}_m}$, $\lambda \notin \mathbb{Z}$, we see $M = \mathbf{IC}(N) = (xd-\lambda)D_{\mathbb{A}^1}\backslash D_{\mathbb{A}^1} \otimes_{\mathbb{C}} \mathbb{C}[\partial_y]$, which equals $\mathbb{C}[x^\lambda] \otimes_{\mathbb{C}} \mathbb{C}[\partial_y]$ as a vector space where $\mathbb{C}[x^\lambda]$ is the $\mathbb{C}$-span of $\{\ldots, x^{\lambda-1}, x^\lambda, x^{\lambda+1}\ldots\}$ infinite in both directions. Therefore, applying multiplication by x and y is surjective. Therefore, we have that $\mathrm{Ext}^1(D_X, M) = M/Mf = 0$. □

The examples include the nodal curve $\mathbb{C}[x(x-1), x^2(x-1)] \subset \mathbb{C}[x]$; in this case $X = X^{sn}$, and the induced map from the intersection of 2 lines in a plane to X is formally locally an isomorphism. In fact, every planar-multicross must be formally locally of this form. It does not include examples like formally locally non-transverse intersections.

More generally, we can globalise to get the following theorem:

Theorem 2.3.13. Given a curve X, there is another curve X^{pl}, whose singularities are all formally locally lines intersecting in a plane, and whose category of D-modules is derived equivalent to that of X.

Therefore there is a zigzag diagram

$$X \twoheadleftarrow X^{sn} \twoheadrightarrow X^{pl},$$

where each arrow is a universal homeomorphism and thus gives a derived equivalence of D-modules and the category is 'nicer' to study in X^{pl}.

Proof. At each singular point $p_i \in X^{sn}$ and U_i such that $p_i \in U_i$, we have $\mathcal{O}(\widehat{(U_i)_{p_i}}) \cong \mathbb{C}[\![x_{i,1}]\!] \oplus \cdots \oplus \mathbb{C}[\![x_{i,n_i}]\!]$. Then by Nakayama's lemma we pick U_i small enough such that $\exists y_{i,1}, \ldots, y_{i,n_i} \subset \mathcal{O}(U_i)$ generate $\mathcal{O}(U_i)$ with $y_{i,j} \equiv x_{i.j} \pmod{\mathfrak{m}_i{}^2}$. This is true because the statement is true in $\mathcal{O}(U_i)_p$ and we can pick U_i small enough that $y_{i,j}$ extend to U_i and generate. Define $X^{pl} := (X^{sn}, \mathcal{O}^{pl})$, where $\mathcal{O}^{pl} \subset \mathcal{O}^{sn}$ is a sheaf of rings, defined by $f|_{U_i} \in \mathcal{O}^{pl}(U_i)$ if $f|_{U_i} \in \mathbb{C}\langle y_{i,1} + \cdots + y_{i,n_i-1}, y_{i,2} + \cdots + (n_i - 2)y_{i,n_i-1} + y_{i,n_i} \rangle \subset \mathcal{O}(U_i)^{sn}$ for all i, the subalgebras generated by these two elements (we are using the matrix at the beginning of this section); along with the open set covering the smooth part, this satisfies the sheaf condition. We need to check that $U_i^{sn} \backslash \{p_i\} \cong U_i^{pl} \backslash \{p_i\}$, *i.e.* $\mathcal{O}(U_i \backslash \{p_i\})$ is generated by $y_{i,1} + \cdots + y_{i,n_i-1}, y_{i,2} + \cdots + (n_i - 2)y_{i,n_i-1} + y_{i,n_i}$ And there is a universal homeomorphism from X^{sn} to X^{pl}. This is true because the singular locus of U_i is just p_i and $\forall q \in U_i \backslash \{p_i\}$, $\exists i$ such that $y'_{i,j}(q) \neq 0$, then by Nakayama's lemma every local neighbourhood in U_i is generated by $y_{i,j}$, so $\mathcal{O}(U_i \backslash \{p_i\})$ is also generated by them. By construction X^{pl} is formally locally n curves meeting transversely in a plane, Zariski locally some of these curves could coincide. □

Fix this?

2.4 Holonomic D-module on isolated quotient singularities

Let $X = V/G$ be the quotient of a smooth variety V by a finite group G acting linearly. In this section, we calculate the image of holonomic D-modules of X under our canonical map $D(D\text{-mod}_X) \to D(\text{REnd}(D_X)\text{-mod})$ in Proposition 2.1.5. We need to calculate $\text{Ext}^{>0}(D_X, M)$. By Lemma 2.3.1, it suffices to show vanishing in a formal neighbourhood of an isolated singularity. Therefore we can assume V is the affine space $\mathbb{A}^n$. In particular, we will see that intermediate extension of the D-module Ω_X can be viewed as an ordinary module over the Grothendieck ring of differential operators Diff(X). The intermediate extension of nontrivial local systems L on $U = X \backslash \{0\}$ where G acts freely away from the origin, have non-vanishing $\text{Ext}^{\dim V - 1}(D_X, IC(L))$. This is completely opposite to the curve case.

Let $\pi : V \to X$ be the quotient map. The strategy is the following: we first study the fundamental exact triangle associated to the intersection cohomology D-module and compute its cone just as we did in Section 2.3 (cf. Lemma 2.4.6). Using this, we can compute $\pi^! \mathbf{IC}(N)$ (cf. Proposition 2.4.7). Finally, using equivariant-adjunction, we can compute $\mathrm{Ext}^i(D_X, \mathbf{IC}(N))$ (cf. Theorem 2.4.8). The results can be globalised, (cf. Remark 2.4.10).

Recall that a pseudo-reflection is an invertible linear transformation $g : V \to V$ such that the order of g is finite and the fixed subspace V^g has codimension 1. The Chevalley–Shephard–Todd theorem [Che55] says:

Theorem 2.4.1. For $x \in V = \mathbb{A}^n$, let $G_x \subset G$ be the stabilizer of x. Then the quotient X is smooth at $\mathrm{Im}(x)$ if and only if G_x is generated by pseudo-reflections.

Suppose that K is the subgroup generated by pseudo-reflections (it is a normal subgroup). Then V/K is smooth by the above theorem. Since $X = V/G = (V/K)/(G/K)$, where V/K is smooth and is isomorphic to V, we can assume without loss of generality that G contains no pseudo-reflections. We can also assume $\dim V \geq 2$ as otherwise we recover the curve case.

For each point in the free locus $\tilde{U}$, the stabilizer is trivial, therefore the condition of Theorem 2.4.1 is satisfied. Thus the quotient of the free locus is smooth. And the singular locus is exactly the image of the complement of the free locus and it has codimension at least 2 by assumption. Let U be the smooth locus, the image of $\tilde{U}$. As G contains no pseudo-reflections, locally this is a smooth covering map, we see that $\pi^! D_U = D_{\tilde{U}}$ and $\pi^! \Omega_U = \Omega_{\tilde{U}}$.

Lemma 2.4.2. $\mathrm{Diff}(X) \cong \mathrm{Diff}(V)^G$

(Apparently is a theorem by [Kan77], we don't really use this any place other than another remark, and our proof is kind of hand wavy, should we just cite this instead?) Ask Travis

Proof. Every differential operator on $X = V/G$ gives a G-invariant differential operator on V_{free}, which by taking principal symbol gives a section of $\mathrm{Sym}^m T_{V_{\text{free}}}$. As G contains no pseudo-reflections, by Hartogs' theorem, these sections extend uniquely to $\mathrm{Sym}^m T_V$, that is, principal symbols of differential operators in $\mathrm{Diff}(V)^G$. Therefore by induction on m, we can conclude the lemma. $\square$

The next lemma was first proved in [ES09, Lemma 2.9] by considering the induction functor Ind from $\mathcal{O}$-modules to right D-modules on any algebraic variety and noting that $\pi_* \circ \mathrm{Ind} = \mathrm{Ind} \circ \pi_*$ for all proper morphisms π (this follows from the two adjunctions Ind $\dashv$ Res and $\pi_* \dashv \pi^!$). Here we give a more direct proof. Note that neither proof uses the assumption that G contains no pseudo-reflections.

Lemma 2.4.3. For $X = V/G$, $D_X = (\pi_* D_V)^G$.

Proof. Choose an embedding $i : X \to Y$, we write π' for $i \circ \pi$. We can compute

$$\begin{aligned}(\pi'_* D_V)^G &= \pi'_\bullet (D_{V\to Y})^G \\ &= \pi'_\bullet (\mathcal{O}(V) \otimes_{\mathcal{O}(Y)} D_Y)^G \\ &= \pi'_\bullet (\mathcal{O}(V)^G \otimes_{\mathcal{O}(Y)} D_Y) \\ &= I_X \backslash D_Y \\ &= i_* D_X.\end{aligned}$$

□

Lemma 2.4.4. If G contains no pseudo-reflections then

$$\Gamma(V, \pi^! D_X)^G \cong \Gamma(V, D_V)^G.$$

Proof. From the previous lemma we can conclude that

$$\mathrm{Hom}((\pi_* D_V)^G, D_X) = \mathrm{Hom}(D_X, D_X) = \mathrm{Diff}(X),$$

by Theorem 2.1.9. The right hand side equals $\mathrm{Diff}(V)^G = \mathrm{Hom}(D_V, D_V)^G$ when G contains no pseudo-reflections. By adjunction, the left hand side also equals

$$\mathrm{Hom}((\pi_* D_V)^G, D_X) \cong \mathrm{Hom}(D_V, \pi^! D_X)^G.$$

Using $\Gamma(V, -) \cong \mathrm{Hom}(D_V, -)$ we conclude the lemma. □

In the following we assume $X = V/G$ has an isolated singularity.

We begin the calculation of the Ext groups. This requires a few steps:

First we generalise [ES09, Lemma 4.3] slightly to include all topological local systems.

Lemma 2.4.5. Let X be an irreducible affine variety of dimension d with a $\mathbb{C}^*$-action having a unique fixed point 0, which is attracting (*i.e.*, X is a cone). Let $U = X\backslash 0$. Assume that U is smooth. Let $j : U \to X$ be the corresponding open embedding and L a $\mathbb{C}^*$-equivariant topological local system on U. make sure this condition is what we need Then, for $m > 0$,

$$\mathrm{Ext}^m(\mathbf{IC}(L), \delta) = \mathrm{Ext}^m(\delta, \mathbf{IC}(\mathbb{D}(L))) = H^{d-m}_{dR}(U, L)^*$$

We include a proof for completeness. Our proof is essentially the same as the proof in [ES09].

Proof. The first equality follows from Verdier duality and that the D-module δ is self dual and $\mathbb{D}(\mathbf{IC}(L)) \cong \mathbf{IC}(\mathbb{D}(L))$ [HTT08, Proposition 3.4.3].

Let $i : 0 \hookrightarrow X$ be the closed embedding. For the second equality we have

$$\begin{aligned}
\operatorname{Ext}^m(\delta, \mathbf{IC}(\mathbb{D}(L))) &\cong \operatorname{Ext}^m(i_*\mathbb{C}, \mathbf{IC}(\mathbb{D}(L))) \\
&\cong \operatorname{Ext}^m(\mathbb{C}, i^! \,\mathbf{IC}(\mathbb{D}(L))) && \text{adjunction} \\
&\cong \operatorname{Ext}^m(\mathbb{D}(i^! \,\mathbf{IC}(\mathbb{D}(L))), \mathbb{C}) && \text{Verdier duality} \\
&\cong \operatorname{Ext}^m(i^*\mathbb{D}\,\mathbf{IC}(\mathbb{D}(L)), \mathbb{C}) \\
&\cong \operatorname{Ext}^m(i^*\,\mathbf{IC}(L), \mathbb{C}) && \text{[HTT08, Proposition 3.4.3]} \\
&\cong H^{-m}(i^*\,\mathbf{IC}(L))^*
\end{aligned}$$

Consider the standard exact triangle

$$\mathbf{IC}(L) \to j_*L \to C \to,$$

where $C \cong i_*i^!\,\mathbf{IC}(L)[1]$. We know that $\mathbf{IC}(L)$ is concentrated in degree 0 and j_*L is concentrated in non-negative degrees. Note that $\mathbf{IC}(L) \to H^0 j_*L$ is injective as $\mathbf{IC}(L)$ is the image of canonical morphism $H^0 j_!L \to H^0 j_*L$ from the adjunction by definition. By considering the long exact sequence we see that C is also concentrated in non-negative degrees and supported at 0. Applying i^* to this triangle, we obtain

$$i^*\,\mathbf{IC}(L) \to i^*j_*L \to i^*C \to .$$

By Kashiwara's theorem, since C is supported at 0, $i^!$ is exact on C. Thus $i^*C = \mathbb{D}i^!\mathbb{D}C$ is also concentrated in non-negative degrees.

We claim that $i^*\,\mathbf{IC}(L)$ is concentrated in negative degrees. Note that as $i^!$ of a D-modules concentrated in degree 0 is concentrated in non-negative degrees [HTT08, Proposition 1.5.14], we only need to prove $i^*\,\mathbf{IC}(L)$ is not concentrated in degree 0. Suppose not, then there is a map $i^*\,\mathbf{IC}(L) \to \delta$ which is non-zero in degree zero. By adjunction, this gives a map $\mathbf{IC}(L) \to \delta$ which is non-zero in degree zero, but this is impossible. Hence by the long exact sequence of homology, we see that $H^{-m}(i^*\,\mathbf{IC}(L)) \cong H^{-m}(i^*j_*(L))$.

Since X is conical and L is equivariant, this is naturally isomorphic to $H^{d-m}(X, j_*(L))$, hence isomorphic to $H^{d-m}(U, L)$. ref?

□

Recall we have the following diagram:

$$\begin{array}{ccccc} \tilde{U} & \xrightarrow{\tilde{j}} & V & \xleftarrow{\tilde{i}} & \{0\} \\ {\scriptstyle \pi|_{\tilde{U}}}\big\downarrow & & {\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle id} \\ U & \xrightarrow[j]{} & X & \xleftarrow[i]{} & \bullet \end{array}$$

With the same setup as the previous lemma, we have:

Lemma 2.4.6. The cone C in the exact triangle $\mathbf{IC}(L) \to j_*L \to C \to$ is isomorphic to $\delta \otimes_{\mathbb{C}} H^{d-1-*}(U, L)$, where d is the dimension of X and δ is the delta module supported at the singularity (*i.e.*, $\delta = i_*(\mathbb{C})$). When L is the trivial local system Ω_U, then $K \cong \delta[1-d]$, and when L is non-trivial, $j_*L = \mathbf{IC}(L)$, *i.e.*, it is a coclean extension.

Proof. The exact triangle $\mathbf{IC}(X) \to j_*L \to C \to$ induces the exact sequence

$$\mathrm{Ext}^m(\delta, j_*L) \to \mathrm{Ext}^m(\delta, C) \to \mathrm{Ext}^{m+1}(\delta, \mathbf{IC}(L)) \to \mathrm{Ext}^{m+1}(\delta, j_*L),$$

and we have

$$\mathrm{Ext}^m(\delta, j_*L) \cong \mathrm{Ext}^m(j^!\delta, L) = 0$$

for all m as $j^!\delta = 0$. Therefore

$$\mathrm{Ext}^m(\delta, C) \cong \mathrm{Ext}^{m+1}(\delta, \mathbf{IC}(L)) \cong H^{d-m-1}_{dR}(U, L)^*.$$

Since C is concentrated at the singularity, it is a direct sum of δ's, so $\mathrm{Ext}^m(\delta, C)$ gives the multiplicity of $\delta[-m]$ in C (for $m \geq 0$). Therefore $K \cong \delta \otimes_{\mathbb{C}} H^{d-1-*}(U, L)$.

Note G is finite and we have the following fibration

$$\begin{array}{ccc} G & \hookrightarrow & \tilde{U} \\ & & \big\downarrow \\ & & U \end{array}$$

Since $\tilde{U} \sim S^{2d-1}$ (homotopic), by Leray–Serre spectral sequence [Wei94, Theorem 5.3.2], we have a spectral sequence

$$H^p(U, H^q(G, \mathbb{C})) \implies H^{p+q}(\tilde{U}).$$

Note that here $H^q(G, \mathbb{C})$ is the cohomology of G as a topological group, not the group cohomology of G. As G is finite, we have $H^q(G, \mathbb{C}) = \mathbb{C}[G]$ concentrated in degree $q = 0$. Hence the spectral sequence degenerates and we have:

$$H^p(U, \mathbb{C}[G]) \cong H^p(S^{2d-1}).$$

Decomposing $\mathbb{C}[G]$ as $\bigoplus_{\text{irreps } V_i} V_i^{\oplus \dim V_i}$ where the 1 dimensional representations gives the local systems. In particular, since $H^p(S^{2d-1}) = 0$ for $0 < p < 2d-1$,we see that $H^p(U, L)$ for $0 < p < 2d-1$ and for all local systems N. And for $H^0(U, L)$, it is $\mathbb{C}$ if L is trivial and 0 is L is non-trivial. This implies, for the trivial local system, $K \cong \delta[1-d]$. For non-trivial local systems, $K = 0$, hence $j_*L = \mathbf{IC}(L)$, *i.e.*, it is a coclean extension. Can compute top cohomology as well, but not necessary. Check with Travis if this is OK.

□

Still keeping the same setup:

Proposition 2.4.7. $\pi^! \mathbf{IC}(X) = \Omega_V$ and $\pi^! \mathbf{IC}(L) = j_* \Omega_{\tilde{U}}$ for L a non-trivial $\mathbb{C}^*$-equivariant simple topological local system.

Proof. For the second statement, we use the fact that it is a coclean extension, the base change formula and the fact that $\pi_1(\tilde{U}) = \{e\}$:

$$\pi^! \mathbf{IC}(L) = \pi^! j_* L = \tilde{j}_* \pi_{|_{\tilde{U}}}{}^! L = \tilde{j}_* \Omega_{\tilde{U}}.$$

For the first statement, note that we have an exact triangle

$$\pi^! \mathbf{IC}(X) \to \pi^! j_* \Omega_U \to \pi^! C \to .$$

Base change formula implies

$$\pi^! j_* \Omega_U \cong \tilde{j}_* \pi_{|_{\tilde{U}}}{}^! \Omega_{\tilde{U}} \cong \tilde{j}_* \Omega_{\tilde{U}}.$$

Let $\tilde{C}$ be the cone of

$$\mathbf{IC}(V) \to \tilde{j}_* \Omega_{\tilde{U}} \to \tilde{C} \to,$$

and $\tilde{C} \cong \tilde{i}_* \tilde{i}^! \mathbf{IC}(V)[1]$, where $\tilde{i}$ is the inclusion map from the origin to V a vector space and $\mathbf{IC}(V) \cong \Omega_V$ (see, for example, [Bra+, Example, page 65]). This implies that $\tilde{C} \cong \tilde{\delta}[1-d]$, where δ is the delta module supported at the preimage of the isolated singularity (*i.e.*, $\delta = \tilde{i}_*(\mathbb{C})$. Here we are abusing the notations of delta modules on X and V). Using the base change formula we see that

$$\pi^! C = \pi^! i_*(\mathbb{C}) \cong \tilde{i}_* \mathrm{id}^!(\mathbb{C}) = \tilde{C}.$$

In other words, we have two exact triangles

$$\pi^! \, \mathbf{IC}(X) \to \tilde{j}_* \Omega_{\tilde{U}} \to \tilde{C} \to$$

and

$$\Omega_V \to \tilde{j}_* \Omega_{\tilde{U}} \to \tilde{C} \to .$$

The two triangles (and in particular the objects $\pi^! \, \mathbf{IC}(X)$ and Ω_V) are determined up to isomorphism by the morphisms $\tilde{j}_* \Omega_{\tilde{U}} \to \tilde{C}$, although not always up to a unique isomorphism. If we can show $\mathrm{Hom}(\tilde{j}_* \Omega_{\tilde{U}}, \tilde{C}) = \mathbb{C}$ and that the two maps of $\tilde{j}_* \Omega_{\tilde{U}} \to \tilde{C}$ are non-zero, then we are done.

We first compute $\mathrm{Hom}(j_* \Omega_{\tilde{U}}, \tilde{C})$. The triangle

$$\Omega_V \to \tilde{j}_* \Omega_{\tilde{U}} \to \tilde{C}$$

induces

$$\mathrm{Ext}^{i-1+d}(\delta, \delta) \to \mathrm{Ext}^i(\tilde{j}_* \Omega_{\tilde{U}}, \delta) \to \mathrm{Ext}^i(\Omega_V, \delta) \to \mathrm{Ext}^{i+d}(\delta, \delta).$$

Note $\mathrm{Ext}^i(\delta, \delta) = \mathbb{C}$ if $i = 0$ and 0 otherwise, and

$$\mathrm{Ext}^i(\mathbf{IC}(V), \delta) \cong \mathrm{Ext}^i(\delta, \mathbf{IC}(V)) \cong H_{dR}^{d-i}(\tilde{U}) \cong H_{dR}^{d-i}(S^{2d-1})$$

for $i > 0$. So $\mathrm{Ext}^i(j_* \Omega_{\tilde{U}}, \delta) = \mathbb{C}$ if $i = d$ or $i = -d + 1$, *i.e.*, $\mathrm{RHom}(j_* \Omega_{\tilde{U}}, \tilde{C}) \cong \mathbb{C} \oplus \mathbb{C}[2d - 1]$. In particular $\mathrm{Hom}(j_* \Omega_{\tilde{U}}, \tilde{C}) = \mathbb{C}$.

We need to show that the two arrows of $j_* \Omega_{\tilde{U}} \to \tilde{C}$ are non-zero. The map $j_* \Omega_{\tilde{U}} \to \tilde{C}$ of the second triangle is non-zero as otherwise it will imply that $\Omega_V \cong j_* \Omega_{\tilde{U}} \oplus \delta[-d]$ which is impossible.

For the first triangle, note that $\mathbf{IC}(X)$ is indecomposable implies that the map $j_* \Omega_U \to C$ is non-zero. As the map $j_* \Omega_{\tilde{U}} \to \tilde{C}$ is the image of $j_* \Omega_U \to C$ under $\pi^! : \mathrm{Hom}(j_* \Omega_U, C) \to \mathrm{Hom}(\pi^! j_* \Omega_U, \pi^! C)$, it suffices to show that $\pi^!$ is an injection between the Hom sets. By adjunction this is identified with $\mathrm{Hom}(j_* \Omega_U, C) \to \mathrm{Hom}(\pi_! \pi^! j_* \Omega_U, C)$, composing with the natural map $\pi_! \pi^! j_* \Omega_U \to j_* \Omega_U$. But

$$\pi_! \pi^! j_* \Omega_U = \pi_! \tilde{j}_* \Omega_{\tilde{U}} = \pi_* \tilde{j}_* \Omega_{\tilde{U}} = j_* \pi_{|_{\tilde{U}} *} \Omega_{\tilde{U}} = j_*(\bigoplus L),$$

summing over all rank one local systems L on U and where the penultimate equality follows from commutativity of the diagram. Under this identification, $\pi_!\pi^! j_*\Omega_U \to j_*\Omega_U$ becomes the projection map $j_*(\bigoplus L) \to j_*\Omega_U$, which is the only non-zero map up to scaling. Since this is a projection map, we obtain that $\pi^!$ is indeed an injection between the Hom sets.

□

For a vector space W with a group G acting on it, denote $W^{G,\perp} := \mathrm{Ker}(W^* \to (W^G)^*) = (W/W^G)^*$.

Theorem 2.4.8. Let L be a $\mathbb{C}^*$-equivariant local system on U, then

$$\mathrm{Ext}^m(D_X, \mathbf{IC}(L)) \cong (\Gamma(V,\delta) \otimes (H_{dR}^{d-m-1}(\tilde{U}, \pi^!\mathbb{D}L)^{G,\perp}))^G$$

for $m > 0$. In particular, we have that $\mathrm{Ext}^\bullet(D_X, \mathbf{IC}(X))$ is concentrated in degree 0.

Proof. Since for any D-module M on X, we have

$$\tilde{j}^*\pi^! M = \pi^!_{|_{\tilde{U}}} j^* M = \tilde{j}^* H^0\pi^! M,$$

where we used j^* and $\pi^!_{|_{\tilde{U}}}$ are exact. We have the following diagram

$$\begin{array}{ccccccc} \pi^! M & \longrightarrow & \tilde{j}_*\tilde{j}^*\pi^! M & \longrightarrow & \tilde{i}_*\tilde{i}^!\pi^! M[1] & \longrightarrow & \\ \uparrow & & \wr\uparrow & & \uparrow & & \\ H^0\pi^! M & \longrightarrow & \tilde{j}_*\tilde{j}^*\pi^! M & \longrightarrow & \tilde{i}_*\tilde{i}^! H^0\pi^! M[1] & \longrightarrow & . \end{array}$$

As $\mathrm{Ext}^{>0}(D_V, \pi^! M) = R^{>0}\Gamma(\mathrm{cone}(H^0\pi^! M \to \pi^! M))$, from the diagram it follows that $\mathrm{Ext}^{>0}(D_V, \pi^! M) = R^{>0}\Gamma(\mathrm{cone}(\tilde{i}_*\tilde{i}^! H^0\pi^! M \to \tilde{i}_*\tilde{i}^!\pi^! M))$.

In the case when $M = \mathbf{IC}(L)$, where N is a holonomic D-module on U, we know that $H^m i^! M \cong H^{d-m}(U, \mathbb{D}L)^*$ for $m \geq 0$ from the fact that $H^{-m} i^* M \cong H_{dR}^{d-m}(U, L)$ (cf. Lemma 2.4.5). Therefore, as $i^!\delta = \mathbb{C}$ and either $\mathbf{IC}(\pi^! L) = H^0\pi^! M$ or $H^0\pi^! M$ is an extension of $\mathbf{IC}(\pi^! L)$ by δ, we see that $H^m\tilde{i}^! H^0\pi^! M \cong H_{dR}^{d-m}(\tilde{U}, \pi^!\mathbb{D}L)^*$ for $m > 0$. Using $H_{dR}^{-m}(U, \mathbb{D}L)^* = H_{dR}^{d-m}(\tilde{U}, \pi^!\mathbb{D}L)^{G^*}$, we see that for $m > 0$,

$$\begin{aligned} \mathrm{Ext}^m(D_X, \mathbf{IC}(L)) &\cong \Gamma H^m \mathrm{cone}(\tilde{i}_*\tilde{i}^! H^0\pi^! M \to \tilde{i}_*\tilde{i}^!\pi^! M)^G \\ &\cong \Gamma H^m \mathrm{cone}(\tilde{i}_*\tilde{i}^! \,\mathbf{IC}(\pi^! L) \to \tilde{i}_*\pi^!\tilde{i}^! \,\mathbf{IC}(L))^G \\ &\cong (\mathrm{Ker}(H_{dR}^{d-m}(\tilde{U}, \pi^!\mathbb{D}L)^* \to H_{dR}^{d-m}(\tilde{U}, \pi^!\mathbb{D}L)^{G^*})[1] \otimes \Gamma(V,\delta))^G \\ &\cong (\Gamma(V,\delta) \otimes (H_{dR}^{d-m-1}(\tilde{U}, \pi^!\mathbb{D}L)^* / H_{dR}^{d-m-1}(U, \mathbb{D}L)^*))^G \\ &\cong (\Gamma(V,\delta) \otimes H_{dR}^{d-m-1}(\tilde{U}, \pi^!\mathbb{D}L)^{G,\perp})^G. \end{aligned}$$

Note that $H^m_{dR}(\tilde{U}, \pi^! L)^* = H^m_{dR}(U, L)^*$ if and only if G acts trivially on $H^m_{dR}(\tilde{U}, \pi^! L)$. Therefore we see that for the trivial local system, $\mathrm{Ext}^{>0}(D_X, \mathbf{IC}(X)) = 0$.

□

Corollary 2.4.9. $\mathrm{Ext}^\bullet(D_X, \mathbf{IC}(X))$ is concentrated in degree 0, and $\mathrm{Ext}^\bullet(D_X, \mathbf{IC}(L))$ is concentrated in degree 0 and $d-1$.

Proof. Since $\tilde{U}$ is homotopic to S^{2d-1}, and $\pi^! L$ is the trivial local system on which G acts by a character, we get that $H^{<d}_{dR}(\tilde{U}, \pi^! \mathbb{D} L)$ is concentrated in degree zero and the G action there is given by the inverse of the character by which G acts on L. This shows that for the trivial local system, the higher Ext groups are always zero; for a non-trivial local system, the higher Ext groups are non-zero in the $d-1$ degree and equals to $\Gamma(V, \delta)_\chi$, where χ is the character by which G is acting on L.

By Proposition 2.4.7, we see that $\mathrm{Ext}^0(D_X, \mathbf{IC}(X)) = (\Omega_V)^G$ for the trivial local system and $\mathrm{Ext}^0(D_X, \mathbf{IC}(L)) = \Gamma(\tilde{U}, \Omega_{\tilde{U}})_\chi$ (where χ is the defining representation for L) for a non-trivial simple topological local system L.

□

Hence we see that the holonomic D-modules can be viewed as a Diff(X) module via our correspondence if at the singularity, either the local system has no monodromy, or it pulls back to a local system on V which is nontrivial at the preimage of the singularity. This includes $\mathbf{IC}(X)$, but excludes $\mathbf{IC}(L)$ for non-trivial simple local systems.I am confused about this sentence and its logic

This is the complete opposite to the case of curves in the Section 2.3.

Remark 2.4.10. The above calculation can be globalised to the following. Suppose now L is a local system on a locally closed subset $Z \subset X = V/G$ where V now is a general variety such that X has isolated singularities. Let $Z_x := Z \cap B_x$, where B_x is a small analytic ball around $x \in X$. Furthermore, let $\widetilde{Z_x}$ be a connected component of $\pi^{-1}(Z_x)$. Then the above calculation also shows that

$$\mathcal{E}\mathrm{xt}^m(D_X, \mathbf{IC}(L))_x \cong H^{d-m-1}\Gamma(\widetilde{Z_x}, \pi^!(\mathbb{D}L))^{G,\perp},$$

for $m > 0$ and $G = \pi_1(Z_x)/\pi_1(\widetilde{Z_x})$, which is 0 if and only if $H^{d-m-1}\Gamma(Z_x, \mathbb{D}L) \hookrightarrow H^{d-m-1}\Gamma(\widetilde{Z_x}, \pi^! \mathbb{D}L)$ is surjective. If L has regular singularity at x, this happens if and only if $L^{\pi_1(Z_x)} \cong L^{\pi_1(\widetilde{Z_x})}$, *i.e.*, $G = \pi_1(Z_x)/\pi_1(\widetilde{Z_x})$ acts trivially on $L^{\pi_1(\widetilde{Z_x})}$.

Therefore:

- if $\dim(Z) \leq 1$, then $\mathrm{Ext}^{>0}(D_X, \mathbf{IC}(L)) = 0$;

- if $\dim(Z) \geq 2$, in general the higher Ext can be complicated. However, if the closure $\widetilde{\overline{Z_x}}$ is a local complete intersection, for each singular point x and X only consists regular singularities, then $\mathrm{Ext}^m(D_X, \mathbf{IC}(L)) = 0$ for $1 \leq m \leq \dim(Z) - 2$ as the link at an isolated complete intersection singularity is $(d-2)$-connected.

Remark 2.4.11. The canonical D-module $M(X)$ of Etingof–Schedler [ES09] is a local enhancement of Poisson homology of X. When $X = V/G$ where G is a finite subgroup of $Sp(V)$ and X has isolated singularities, $M(X)$ is isomorphic to a direct sum of intermediate extensions of trivial local systems on each stratum, see [ES09, Corollary 4.16]. We deduce that $M(X)$ can be seen an ordinary D-module over $\mathrm{Diff}(X)$. This includes the Du Val case.

We can also do a similar calculation to deduce the structure of $\mathrm{Ext}^\bullet(D_X, D_X)$:

Proposition 2.4.12. The cohomology of the DG algebra $\mathrm{REnd}(D_X)$ is given by

$$\mathrm{Ext}^\bullet(D_X, D_X) \cong \mathrm{Diff}(V)^G \bigoplus (\langle (1-g) \cdot \Gamma(V, \delta) \rangle_{g \in G} \otimes \Gamma(V, \delta))^G [1-d].$$

Proof. Note we have the diagram

$$\begin{array}{ccccccc}
\pi^! D_X & \longrightarrow & \tilde{j}_* \tilde{j}^* \pi^! D_X & \longrightarrow & \tilde{i}_* \tilde{i}^! \pi^! D_X[1] & \longrightarrow & \\
\uparrow & & \wr\uparrow & & \uparrow & & \\
H^0 \pi^! D_X & \longrightarrow & \tilde{j}_* \tilde{j}^* \pi^! D_X & \longrightarrow & \tilde{i}_* \tilde{i}^! H^0 \pi^! D_X[1] & \longrightarrow & .
\end{array}$$

Also, we have $H^0 \pi^! D_X \cong D_V$. Indeed, as $D_V \cong H^0 j_* j^* H^0 \pi^! D_X$, we have by adjunction, a map $H^0 \pi^! D_X \to D_V$. This map is injective as the kernel is concentrated at the origin and $\mathrm{Hom}(\delta, H^0 \pi^! D_X) = \mathrm{Hom}(\delta, D_X) = \mathrm{Hom}(\delta, D_V)^G = 0$. As $\mathrm{Ext}^1(\delta, M) = 0$ and D_V is indecomposable, we see this map has to be surjective too.

Furthermore,

$$\begin{aligned}
i^! D_X &\cong \mathrm{RHom}(\mathbb{C}, i^! D_X) \\
&\cong \mathrm{RHom}(\delta, (\pi_* D_V)^G) \\
&\cong \mathrm{RHom}(\mathbb{D} D_V, \pi^! \delta)^G \\
&\cong \mathrm{RHom}(\pi^* \delta, D_V)^G \\
&\cong (i^! D_V)^G \\
&\cong \Gamma(V, \delta)^G [-d].
\end{aligned}$$

Therefore,

$$\begin{aligned}\operatorname{Ext}^m(D_X, D_X) &\cong \Gamma H^m \operatorname{cone}(H^0\pi^! D_X \to \pi^! D_X)^G \\ &\cong \Gamma H^m \operatorname{cone}(\tilde{i}_* \tilde{i}^! H^0 \pi^! D_X \to \tilde{i}_* \tilde{i}^! \pi^! D_X)^G \\ &\cong \Gamma H^m \operatorname{cone}(\tilde{i}_* \tilde{i}^! D_V \to \tilde{i}_* \pi^! i^! D_X)^G \\ &\cong H^{m-d}(\operatorname{Ker}(\Gamma(V,\delta) \to \Gamma(V,\delta)^G)[1] \otimes \Gamma(V,\delta))^G \\ &\cong H^{m-d+1}(\langle(1-g)\cdot\Gamma(V,\delta)\rangle_{g\in G} \otimes \Gamma(V,\delta))^G,\end{aligned}$$

for $m > 0$. Adding $\operatorname{Ext}^0(D_X, D_X) \cong \operatorname{Diff}(X)$ yields the result. □

Remark 2.4.13. Note that $\pi^! D_X \not\cong D_V$ (but $H^0\pi^! D_X \cong D_V$). Indeed suppose $\pi^! D_X \cong D_V$, then

$$\operatorname{Ext}^\bullet(\pi^! D_X, \pi^! D_X)^G \cong \operatorname{Ext}^\bullet(\pi_* D_V, D_X)^G \cong \operatorname{Ext}^\bullet(D_X, D_X)$$

which is not concentrated in degree zero, but $\operatorname{REnd}(D_V)^G$ is.

Remark 2.4.14. The proposition implies that any Kleinian singularity X has non-vanishing $\operatorname{Ext}^1(D_X, D_X)$, hence they are not cuspidal.

Remark 2.4.15. For $M = \mathbf{IC}(L_\chi)$ where L_χ is a simple non-trivial topological local system, the

$$\operatorname{Ext}^{d-1}(D_X, D_X) \times \operatorname{Ext}^0(D_X, \mathbf{IC}(L_\chi)) \to \operatorname{Ext}^{d-1}(D_X, \mathbf{IC}(L_\chi))$$

action becomes

$$(\langle(1-g)\cdot\Gamma(V,\delta)\rangle_{g\in G} \otimes \Gamma(V,\delta))^G \times \Gamma(V, \mathcal{O}_{\tilde{U}})_\chi \to (\Gamma(V,\delta) \otimes H^{-d}_{dR}(\tilde{U}, \pi^! \mathbb{D} L_\chi)^{G,\perp})^G.$$

Note we have $H^{-d}_{dR}(\tilde{U}, \pi^! \mathbb{D} L_\chi)^{G,\perp} = \chi$. On the subrepresentation $\chi \subset \Gamma(\delta)^{G,\perp}$, this action is induced by applying distributions to functions $\chi \otimes \mathcal{O}_{\tilde{U}} \to \chi$, and the action is zero elsewhere.

Remark 2.4.16. We conjecture that it might be possible to unify Section 2.3 and Section 2.4. In some sense, both sections are dealing with computing a cone of the form $H^0\pi^! M \to \pi^! M$ on C for a finite map $\pi : Y \to X$, then reducing to X. In Section 2.3, the map was the normalisation map; in this section, the map is a quotient by a finite group G and the reduction process is taking G-invariants.

Chapter 3

Hochschild–de Rham Homology

3.1 Hochschild–de Rham Homology

Assume throughout this chapter that X is affine and $\mathcal{O}_\hbar(X)$ is a quantisation of $\mathcal{O}(X)$. Recall from the introduction that

$$M(X) := (\mathrm{Ham}_X)\backslash D_X.$$

Because of the bi-differential operator assumption on our star product, we can also define the following quantised version of $M(X)$:

Definition 3.1.1. Let $X \to V$ be a closed embedding into a smooth affine variety. Define

$$M_\hbar(X) := (\mathrm{Ham}_{\hbar,X})\backslash D_X[\![\hbar]\!],$$

where D_X is viewed as before, $D_X[\![\hbar]\!]$ is viewed as a $D_V[\![\hbar]\!]$-module and the (right) submodule $(\mathrm{Ham}_{\hbar,X})$ is spanned by 'quantum Hamiltonian operators' $\xi_{\hbar,f}$, where for $f \in \mathcal{O}_\hbar(X)$, $\xi_{\hbar,f} \in \mathrm{Diff}(X)[\![\hbar]\!]$ is defined by

$$\xi_{\hbar,f}(g) = \frac{1}{\hbar}[f,g]_\star,$$

where $g \in \mathcal{O}(X)$ and $[\cdot,\cdot]_\star$ is given by the commutator of $\star$. It acts on the left on $D_X[\![\hbar]\!]$ since $\mathrm{Diff}(X)$ acts on the left on D_X (by endomorphisms). This makes sense because $\star$ is commutative mod $\hbar$.

It is possible to define a suitable category of formal families of D-modules on X and then $M_\hbar(X)$ is an object there which is independent of the embedding.

Remark 3.1.2. In fact we may define a relative version of $M_\hbar(X)$. Let $\phi : X \to Y$

be a morphism of affine varieties and $\mathcal{O}_\hbar(X)$ a deformation quantisation on X. Define

$$M_\hbar(X,\phi) := \frac{D_X[\![\hbar]\!]}{\mathrm{ad}_\star(\phi^*(\mathcal{O}(Y)))D_X[\![\hbar]\!]},$$

where $\phi^*(\mathcal{O}(Y))$ is realised as a subalgebra of $\mathcal{O}_\hbar(X)$.

Clearly, when ϕ is the identity map, we arrive at our original definition of $M_\hbar(X)$.

We now relate our D-module $M_\hbar(X)$ to Hochschild homology of the quantisation. Let $p : X \to \operatorname{Spec}\mathbb{C}$ and M a $D[\![\hbar]\!]$-module on X. Fix a closed embedding $i : X \to V$ into a smooth affine variety V with the defining ideal I_X, then denote $p_*M = M \otimes^{\mathbb{L}}_{D_V[\![\hbar]\!]} \mathcal{O}(V)[\![\hbar]\!]$ the D-module-theoretic pushforward for formal families.

Notice that $\mathbf{HH}_0(\mathcal{O}_\hbar(X))$ is a $\mathbb{C}[\![\hbar]\!]$-module, the following lemma realises $H^0 p_* M_\hbar(X)$ as a submodule of $\mathbf{HH}_0(\mathcal{O}_\hbar(X))$.

Lemma 3.1.3.

$$H^0 p_* M_\hbar(X) \cong \hbar^{-1}[\mathcal{O}_\hbar(X), \mathcal{O}_\hbar(X)]\backslash\mathcal{O}_\hbar(X) \cong \hbar \cdot \mathbf{HH}_0(\mathcal{O}_\hbar(X)) \subset \mathbf{HH}_0(\mathcal{O}_\hbar(X)),$$

and

$$H^0 p_* M_\hbar(X)[\hbar^{-1}] \cong \mathbf{HH}_0(\mathcal{O}_\hbar(X)[\hbar^{-1}]).$$

Proof. For the first statement:

$$\begin{aligned}
H^0 p_* M_\hbar(X) =& M_\hbar(X) \otimes_{D_V[\![\hbar]\!]} \mathcal{O}(V)[\![\hbar]\!] \\
\cong& \operatorname{Ham}_{\hbar,X} \backslash D_X[\![\hbar]\!] \otimes_{D_V[\![\hbar]\!]} \mathcal{O}(V)[\![\hbar]\!] \\
\cong& \frac{1}{\hbar}[\mathcal{O}(X)[\![\hbar]\!], \mathcal{O}(X)[\![\hbar]\!]]_\star \backslash \mathcal{O}(X)[\![\hbar]\!] \\
\cong& [\mathcal{O}(X)[\![\hbar]\!], \mathcal{O}(X)[\![\hbar]\!]]_\star \backslash \hbar\mathcal{O}(X)[\![\hbar]\!] \\
\cong& \hbar \cdot \mathbf{HH}_0(\mathcal{O}_\hbar(X)),
\end{aligned}$$

where in the penultimate step we used the fact that the multiplication map $\mathcal{O}_\hbar(X) \xrightarrow{\hbar\cdot} \hbar\mathcal{O}(X)[\![\hbar]\!]$ is an isomorphism of $\mathbb{C}[\![\hbar]\!]$-modules. The second statement also follows.

□

We now proceed to define the *Hochschild–de Rham* homology, following the idea of Poisson-de Rham homology in [ES09].

Definition 3.1.4. We define

$$\mathbf{HH}_i^{dR}(\mathcal{O}_\hbar(X)) := H^{-i} p_* M_\hbar(X).$$

Note that $M_\hbar(X)$ has a natural decreasing filtration

$$\cdots \subset \hbar^2 M_\hbar(X) \subset \hbar M_\hbar(X) \subset M_\hbar(X),$$

with $F^i M_\hbar(X) := \hbar^i M_\hbar(X)$.

Recall that given any filtration $F^\bullet$ of M, we can topologise M by taking cosets $m+F^i$ for all $m \in M$ and $i \in I$ as basic open sets.

Definition 3.1.5. We say M is *Hausdorff* if M is Hausdorff in this topology. This is equivalent to $\{0\}$ is closed in M. The closure of $\{0\}$ is $\bigcap_i F^i M$, thus Hausdorff is equivalent to $\bigcap_i F^i M = 0$. Assuming M is Hausdorff, we say M is *complete* if every Cauchy sequence in M has a limit.

An equivalent formulation of the above is if we look at the map $\iota : M \to \hat{M}$ from M to its completion, ι is an injection (respectively, isomorphism) if and only if M is Hausdorff (respectively, complete). See [AM69, Chapter 10].

Remark 3.1.6. (Different versions of the Hochschild–de Rham homology)

It is then perhaps more natural to define $M_\hbar(X)$ as the quotient of $D_X[\![\hbar]\!]$ by $\overline{(\mathrm{Ham}_{\hbar,X})}$ the closure of the submodule $(\mathrm{Ham}_{\hbar,X})$ with respect to the $\hbar$-adic topology. Call this version $\overline{M_\hbar(X)}$.

It follows that:

Lemma 3.1.7.

$$\overline{M_\hbar(X)} = M_\hbar(X) / \bigcap_m \hbar^m M_\hbar(X).$$

Proof. Indeed, this is because

$$\begin{aligned} P \in \overline{(\mathrm{Ham}_{\hbar,X})} &\iff \text{for all } m \geq 0, \exists P_m \in (\mathrm{Ham}_{\hbar,X}) \text{ such that } \hbar^m | (P - P_m), \\ &\iff \text{for all } m \geq 0, \mathrm{Im}(P) \text{ under } \overline{(\mathrm{Ham}_{\hbar,X})} \hookrightarrow D_X[\![\hbar]\!] \twoheadrightarrow M_\hbar(X) \\ &\quad\ \ \text{satisfies } \hbar^m | \, \mathrm{Im}(P). \qquad (*) \end{aligned}$$

□

The advantage is that $\overline{M_\hbar(X)}$ is Hausdorff and complete (as quotient of a complete space by a closed subspace is complete). However, we claim that this is unnecessary.

Proposition 3.1.8. Let X be a variety and $\mathcal{O}_\hbar(X)$ a quantisation of X, then:

1. If S is a finite set of topological generators of $\mathcal{O}_\hbar(X)$, then

$$M_\hbar(X) \cong \mathrm{ad}_\star(S)D_X[\![\hbar]\!]\backslash D_X[\![\hbar]\!].$$

2. In particular, we can take $S = \{\hbar\} \cup S'$ where S' is any finite set of generators of $\mathcal{O}(X)$. This implies $\overline{M_\hbar(X)} \cong M_\hbar(X)$, that is, $M_\hbar(X)$ is already Hausdorff and complete.

3. If T is contains $\mathcal{O}(X)$, then $M_\hbar(X) \cong \mathrm{ad}_\star(T)D_X[\![\hbar]\!]\backslash D_X[\![\hbar]\!]$. In particular,

$$M_\hbar(X) \cong \mathrm{ad}_\star(\mathcal{O}(X))D_X[\![\hbar]\!]\backslash D_X[\![\hbar]\!].$$

Here and below, by topological generators we mean that the closure of the subring generated by them is the whole topological ring.

Proof. If S is a finite set of topological generators of $\mathcal{O}_\hbar(X)$, then we have

$$\mathrm{ad}_\star(S)D_X[\![\hbar]\!] \hookrightarrow \mathrm{ad}_\star(\langle S\rangle_\star)D_X[\![\hbar]\!] \hookrightarrow \overline{\mathrm{ad}_\star(\langle S\rangle_\star)}D_X[\![\hbar]\!] = \mathrm{ad}_\star(\mathcal{O}_\hbar(X))D_X[\![\hbar]\!],$$

where the notation $\langle S\rangle_\star$ means the subring generated by S under the $\star$ product in $\mathcal{O}_\hbar(X)$, and the last equality follows that S is a set of topological generators and that $\mathrm{ad}_\star$ is $\hbar$-adic continuous. This induces

$$\frac{D_X[\![\hbar]\!]}{\mathrm{ad}_\star(S)D_X[\![\hbar]\!]} \xrightarrow{(i)} \frac{D_X[\![\hbar]\!]}{\mathrm{ad}_\star(\langle S\rangle_\star)D_X[\![\hbar]\!]} \xrightarrow{(ii)} \frac{D_X[\![\hbar]\!]}{\overline{\mathrm{ad}_\star(\langle S\rangle_\star)D_X[\![\hbar]\!]}} \xrightarrow{\sim} M_\hbar(X).$$

First note that

$$\begin{aligned}[f\star g, v]_\star =& f\star g\star v - v\star f\star g\\ =& f\star(g\star v) - (g\star v)\star f\\ &+ g\star(v\star f) - (v\star f)\star g\\ =& [f, g\star v]_\star + [g, v\star f]_\star,\end{aligned}$$

for any $f, g, v \in \mathcal{O}_\hbar(X)$. This implies that

$$\mathrm{ad}_\star(f\star g)(-) = \mathrm{ad}_\star(f)(g\star -) + \mathrm{ad}_\star(g)(-\star f).$$

As we assumed that all of our star products are differential star products, this implies that if S is a set of generators of B, then $\mathrm{ad}_\star(\langle S\rangle_\star)D_X[\![\hbar]\!] = \mathrm{ad}_\star(S)D_X[\![\hbar]\!]$. Hence the first map (i) is an isomorphism.

As S is a finite set, there is a surjection $D_X[\![\hbar]\!]^n \twoheadrightarrow \mathrm{ad}_\star(\langle S\rangle_\star)D_X[\![\hbar]\!]$. Then the fact that the sum of closed maps is close implies $\mathrm{ad}_\star(\langle S\rangle_\star)D_X[\![\hbar]\!]$ is closed. Hence the second map (ii) is also an isomorphism. This proves (1).

It remains to establish a finite set of topological generators. As X is a variety, $\mathcal{O}(X)$ is finitely generated under its usual $(\cdot)$ multiplication structure. Take $S' := \{s_1, \ldots, s_n\}$ to be a finite set of generators of $\mathcal{O}(X)$, then we claim the set $S := \{\hbar, s_1, \ldots, s_n\}$ topologically generates $\mathcal{O}_\hbar(X)$. Indeed, take $f = \sum_i f_i \hbar^i \in \mathcal{O}_\hbar(X)$. We know that f_0 is generated by $s_i \in S'$ under the usual product, write this as $f_0 = g_0(S')$, where g_0 consists of only $(\cdot)$ multiplication and addition. Switching to the $\star$ product will yield $g_{0\star}(S') - f_0 \in \hbar\mathcal{O}_\hbar(X)$, where $g_{0\star}$ is g_0 where all $(\cdot)$ multiplications are replaced with $(\star)$ multiplications. Denote the coefficient of $\hbar$ in $g_{0\star}(S') - f_0 \in \hbar\mathcal{O}_\hbar(X)$ to be c_1, then we can find g_1 such that $f_1 - c_1 = g_1(S')$ and $(g_{0\star}(S') + g_{1\star}(S')\hbar) - (f_0 + f_1\hbar) \in \hbar^2\mathcal{O}_\hbar(X)$. Inductively we see that f is in the closure of S. This proves (2).

If $\mathcal{O}(X) \subseteq T$, then

$$\mathrm{ad}_\star(T)D_X[\![\hbar]\!] \supseteq \mathrm{ad}_\star(\mathcal{O}(X))D_X[\![\hbar]\!] \supseteq \mathrm{ad}_\star(\langle \mathcal{O}(X)\rangle_\star)D_X[\![\hbar]\!] \supseteq \mathrm{ad}_\star(\mathcal{O}_\hbar(X))D_X[\![\hbar]\!].$$

Hence $\mathrm{ad}_\star(T)D_X[\![\hbar]\!] = \mathrm{ad}_\star(\mathcal{O}_\hbar(X))D_X[\![\hbar]\!]$. This proves (3).

□

It is perhaps even more natural to define $\widehat{\mathbf{HH}}_i^{dR}(\mathcal{O}_\hbar(X))$ to be the completion of $\mathbf{HH}_i^{dR}(\mathcal{O}_\hbar(X))$ with respect to the $\hbar$-adic topology. However, we claim that this is also unnecessary.

Proposition 3.1.9. $\mathbf{HH}_i^{dR}(\mathcal{O}_\hbar(X))$ is complete with respect to the $\hbar$-adic topology.

Proof. We already know $M_\hbar(X)$ is complete. Embedding X into a smooth affine V, we can resolve $\mathcal{O}(V)$ as a D_V-module by the de Rham complex. Tensoring with $M_\hbar(X)$ we have:

$$0 \to M_\hbar(X) \otimes_{\mathcal{O}(V)[\![\hbar]\!]} \bigwedge^n \Theta_V[\![\hbar]\!] \to \cdots \to M_\hbar(X) \otimes_{\mathcal{O}(V)[\![\hbar]\!]} \bigwedge^0 \Theta_V[\![\hbar]\!] \to M_\hbar(X) \to 0,$$

where Θ_V is the sheaf of vector fields and $n = \dim V$. The differential is given by

$$\begin{aligned} d(m \otimes \theta_1 \wedge \cdots \wedge \theta_k) = & \sum_i (-1)^{i+1} m\theta_i \otimes \theta_1 \wedge \cdots \wedge \hat{\theta}_i \wedge \cdots \wedge \theta_k \\ & + \sum_{i<j} (-1)^{i+j} m \otimes [\theta_i, \theta_j] \wedge \theta_1 \wedge \cdots \wedge \hat{\theta}_i \wedge \cdots \wedge \hat{\theta}_j \cdots \wedge \theta_k. \end{aligned}$$

As the differentials are continuous, the kernels are closed, hence complete. The images are also closed since the maps are (alternating) sums of closed maps. This proves the claim. $\square$

Remark 3.1.10. In fact the same argument can be used to improve [ES17, Theorem 3.4]; we can remove the requirement of finite-dimensional representations being continuous. The proof there takes the closure $\overline{[A_\hbar, A_\hbar]}$, but by deploying the same technique of Proposition 3.1.8, one can show $[A_\hbar, A_\hbar]$ is already closed.

We now relate our $M_\hbar(X)$ to $M(X)$.

Theorem 3.1.11. There is a canonical surjection

$$M(X)[\hbar] \twoheadrightarrow \operatorname{gr}_{\hbar-\mathrm{adic}} M_\hbar(X). \tag{†}$$

Proof. For $W \subset V$ filtered vector spaces, there is always a canonical surjection $\operatorname{gr} W \backslash \operatorname{gr} V \twoheadrightarrow \operatorname{gr}(W \backslash V)$, therefore we have a surjection

$$\operatorname{gr}_\hbar(\mathrm{Ham}_{\hbar,X}) \backslash \operatorname{gr}_\hbar D_X[\![\hbar]\!] \twoheadrightarrow \operatorname{gr}_\hbar((\mathrm{Ham}_{\hbar,X}) \backslash D_X[\![\hbar]\!]).$$

In general for any filtered subset $R \subset V$ we always have $(\operatorname{gr} R) \subset \operatorname{gr}(R)$, where (R) means the smallest submodule containing R, therefore $(\mathrm{Ham}_X)[\hbar] \subset \operatorname{gr}_\hbar(\mathrm{Ham}_{\hbar,X})$ and

$$M(X)[\hbar] \twoheadrightarrow \operatorname{gr}_\hbar(\mathrm{Ham}_{\hbar,X}) \backslash \operatorname{gr}_\hbar D_X[\![\hbar]\!].$$

Composing the two surjections completes the proof. $\square$

Remark 3.1.12. By taking the underived direct image to a point we recover

$$\mathbf{HP}_0^{dR}(X)[\hbar] \twoheadrightarrow \operatorname{gr}_\hbar \mathbf{HH}_0^{dR}(\mathcal{O}_\hbar(X)) \cong \hbar \operatorname{gr}_\hbar \mathbf{HH}_0(\mathcal{O}_\hbar(X)).$$

When X is smooth symplectic, it is known in [ES09, Example 2.6] that $M(X) \cong \Omega_X$. We also get the same statement for $M_\hbar(X)$:

Proposition 3.1.13. If X is smooth symplectic, then $M_\hbar(X) \cong \Omega_X[\![\hbar]\!]$.

Before we prove this proposition, we need a basic well-known lemma:

Lemma 3.1.14. Let M and N be two modules with decreasing filtrations $F^\bullet$ and $G^\bullet$ respectively, assume they are Hausdorff. Then if $f : M \to N$ is a filtered map such that $\operatorname{gr} f : \operatorname{gr} M \to \operatorname{gr} N$ is an injection of graded modules, then f is an injection. If moreover M is complete with respect to $F^\bullet$, then if $f : M \to N$ is a filtered map such that $\operatorname{gr} f : \operatorname{gr} M \to \operatorname{gr} N$ is a surjection of graded modules, then f is a surjection.

We include a proof since we couldn't find one in the literature.

Proof. We deal with injectivity first. Let $m \in M$ be a non-zero element, let k be the number such that $m \in F^k M$ and $m \notin F^{k+1}M$. Such k exists because $F^\bullet$ is Hausdorff. Let $\overline{m}$ denote the image of m in $\operatorname{gr} M$, then as $\overline{m}$ is non-zero, $\operatorname{gr} f(\overline{m}) \neq 0$, which implies $\overline{f(m)} \neq 0$, hence $f(m) \neq 0$.

For surjectivity, assume $n \in N$ and let k be the number such that $n \in G^k N$ and $n \notin G^{k+1}N$. As $\operatorname{gr} f$ is surjective, there is an $m \in F^k M$ such that $\operatorname{gr} f(\overline{m}) = \overline{n}$. Then $n_1 := f(m) - n$ is in $G^{k+1}N$. Now do the same process to n_1, we can find an element $m_1 \in F^{k+1}M$ such that $f(m_1) - n_1 \in G^{k+2}N$. Continuing this process, we can write $n = f(m + m_1 + \dots)$. As M is complete, the sum makes sense.

□

Proof of Proposition. Let $\pi \in \bigwedge^2 TX$ be the Poisson structure. We first construct a map $\phi : M_\hbar(X) \to \Omega_X[\![\hbar]\!]$. As (X, π) is smooth symplectic, Dolgushev showed that for any $\pi_\hbar = \sum_i \pi_i \hbar^i$ with $\pi_1 = \pi$, there exists a formal power series of top degree forms $\omega_\hbar = \sum_i \omega_i \hbar^i \in \Omega^{\dim X}(X)[\![\hbar]\!]$, starting with a nowhere vanishing form ω_0, such that $\mathcal{L}_{\pi_\hbar}\omega_\hbar = 0$ ([Dol09, Proposition 3.1]). Choose $\pi_\hbar$ to be the canonical formal Poisson structure according to Kontsevich's Formality Theorem (see [Bel+16, Chapter II, Theorem 2.3.13] and the references therein). Let $M_\hbar(X) \to \Omega_X[\![\hbar]\!]$ be the map that sends 1 to $\omega_\hbar$, this is well-defined because the equation $\mathcal{L}_{\pi_\hbar}\omega_\hbar = 0$ implies that $di_{\pi_\hbar}\omega_\hbar = 0$ and hence

$$\operatorname{ad}_\star f \cdot \omega_\hbar := \mathcal{L}_{(\operatorname{ad}_\star f)}\omega_\hbar = di_{\pi_\hbar(df,-)}\omega_\hbar = d(i_{\pi_\hbar}\omega_\hbar \wedge df) = 0.$$

Since Ω is irreducible and ϕ is non-zero mod $\hbar$, [ES09] showed that the map $D_X[\![\hbar]\!] \twoheadrightarrow M_\hbar(X) \to \Omega[\![\hbar]\!]$ is surjective mod $\hbar$. Because both modules are complete, one can show inductively on degree of $\hbar$ that this map is surjective. As this surjection factorises through ϕ, ϕ is also surjective. To show it is injective, we consider the associated graded map $\operatorname{gr}_\hbar M_\hbar(X) \twoheadrightarrow \operatorname{gr}_\hbar \Omega_X[\![\hbar]\!]$. By the aforementioned result in [ES09], $\operatorname{gr}_\hbar \Omega_X[\![\hbar]\!] \cong M(X)[\hbar]$. But by Theorem 3.1.11, there is also a surjection $M(X)[\hbar] \twoheadrightarrow \operatorname{gr}_\hbar M_\hbar(X)$. As the modules are finitely-generated, and surjective endomorphisms of finitely-generated modules over Noetherian rings are automorphisms, they must be isomorphic. Hence $\Omega_X[\hbar] \cong \operatorname{gr}_\hbar M_\hbar(X)$, and by the previous lemma, we deduce they are isomorphic on the filtered level.

□

Lemma 3.1.15.
$$\mathbf{HH}^{dR}_\bullet(\mathcal{O}_\hbar(X)) \cong H^{\dim X - \bullet}(X, \mathbb{C}[\![\hbar]\!]),$$
for X smooth symplectic.

Proof. This follows from taking hypercohomology of the D-module theoretic derived pushforward from X to a point of both sides of the equation in the previous proposition and using the de Rham resolution of Ω_X to resolve the right hand side. Then isomorphism follows from Grothendieck's theorem on de Rham cohomology [Gro66]. □

Corollary 3.1.16.

$$\mathbf{HH}^{dR}_\bullet(\mathcal{O}_\hbar(X)[\hbar^{-1}]) \cong \mathbf{HH}_\bullet(\mathcal{O}_\hbar(X)[\hbar^{-1}]),$$

for X smooth symplectic.

Proof. By results of Nest–Tsygan and Brylinski ([NT95, Theorem A2.1], [Bry88]),

$$\mathbf{HH}_\bullet(\mathcal{O}_\hbar(X)[\hbar^{-1}]) \cong H^{\dim X-\bullet}(X, \mathbb{C}(\!(\hbar)\!)),$$

now the corollary follows from the previous lemma. □

3.2 Symplectic Resolutions

It follows that if the canonical surjection equation (†) in Theorem 3.1.11 is an isomorphism, we have that

$$\mathbf{HP}^{dR}_\bullet(X)[\hbar] \cong H^\bullet(M(X)[\hbar] \otimes^{\mathbb{L}}_{D_V[\hbar]} \mathcal{O}(V)[\hbar]) \cong H^\bullet(\operatorname{gr}_\hbar M_\hbar(X) \otimes^{\mathbb{L}}_{D_V[\hbar]} \mathcal{O}(V)[\hbar]).$$

Note that by [Wei94, Section 5.4] for spectral sequence of a filtration, the following spectral sequence converges

$$E_1^{p,q} = H^{p+q}(\operatorname{gr}_\hbar(M_\hbar(X) \otimes^{\mathbb{L}}_{D_V[\![\hbar]\!]} \mathcal{O}(V)[\![\hbar]\!])) \Rightarrow_p \operatorname{gr}_\hbar H^{p+q}(M_\hbar(X) \otimes^{\mathbb{L}}_{D_V[\![\hbar]\!]} \mathcal{O}(V)[\![\hbar]\!]).$$

Here, $M_\hbar(X) \otimes^{\mathbb{L}}_{D_V[\![\hbar]\!]} \mathcal{O}(V)[\![\hbar]\!]$ is filtered, see for example [Gal18, Lemma 3.3]. And

$$\operatorname{gr}_\hbar(M_\hbar(X) \otimes^{\mathbb{L}}_{D_V[\![\hbar]\!]} \mathcal{O}(V)[\![\hbar]\!]) \cong (\operatorname{gr}_\hbar M_\hbar(X)) \otimes^{\mathbb{L}}_{D_V[\hbar]} \mathcal{O}(V)[\hbar],$$

[Gal18, Page 11]. The reference is for R commutative.

Hence we see that:

Lemma 3.2.1. If the canonical surjection equation (†) in Theorem 3.1.11 is an isomorphism

$$\operatorname{gr}_\hbar M_\hbar(X) \cong M(X)[\![\hbar]\!],$$

then we have a Brylinski-type spectral sequence

$$\mathbf{HP}^{dR}_\bullet(X)[\hbar] \Rightarrow_p \operatorname{gr}_\hbar \mathbf{HH}^{dR}_\bullet(\mathcal{O}_\hbar(X)).$$

We already know that $M(X)[\hbar] \cong \operatorname{gr}_\hbar M_\hbar(X)$ when X is smooth symplectic. And the spectral sequence degenerates as it can be identified with the classical Brylinski spectral sequence.

We want to generalise the situation from symplectic smooth to *symplectic resolution*. Before proving the next theorem, we first recall the definition:

Definition 3.2.2. A *symplectic singularity* is an irreducible algebraic variety X over $\mathbb{C}$ equipped with a smooth projective resolution $\rho : \tilde{X} \to X$ such that:

- X is normal;
- there is a non-degenerate symplectic form $\omega_{reg} \in H^0(X^{reg}, \Omega^2)$ on the smooth locus $X^{reg} \subset X$;
- for some ($\iff$ every) projective resolution $\rho : \tilde{X} \to X$, $\rho^*\omega_{reg}$ extends to $\tilde{X}$ (possibly degenerate).

We say $\rho : \tilde{X} \to X$ as above is a *symplectic resolution* if $\rho^*\omega_{reg}$ is non-degenerate (*i.e.*, symplectic); we say X is *conical* if

1. $\mathcal{O}(X) = \bigoplus_{n \geq 0} \mathcal{O}(X)_n$ with $\mathcal{O}(X)_0 = \mathbb{C}$,
2. the Poisson bracket is homogeneous of degree $-l$, for some $l > 0$.

In this case, the resolution $\rho : \tilde{X} \to X$ lifts the $\mathbb{C}^*$ action to $\tilde{X}$ making ρ equivariant. See [GK04, Lemma 5.3].

In general, we always have that $\rho : \tilde{X} \to X$ is a symplectic resolution $\implies$ X is a symplectic singularity $\implies$ X has finitely many symplectic leaves. See [Kal03].

Theorem 3.2.3. Let $\rho : \tilde{X} \twoheadrightarrow X$ be a projective symplectic resolution such that

- $\rho_*\Omega_{\tilde{X}} \cong M(X)$,
- X has locally conical singularities.

Assume the quantisation $\mathcal{O}_\hbar(X)$ extends to a quantisation $\mathcal{O}_\hbar(\mathfrak{X})$ on a (one-parameter) formal Poisson smoothing $\mathfrak{X}$ of X. Then $M(X)[\hbar] \cong \operatorname{gr} M_\hbar(X)$ as graded modules. Moreover this can be strengthened to $M_h(X) \to M(X)[\![\hbar]\!] \cong \rho_*\Omega_{\tilde{X}}[\![\hbar]\!]$ is an isomorphism as filtered modules.

Here and below, by X has locally conical singularities we mean that for every point $x \in X$, there is an open neighbourhood U containing x that is conical.

Remark 3.2.4. Note the two conditions in the theorem are two separate conjectures:

- The first condition $\rho_*\Omega_{\tilde{X}} \cong M(X)$ has been conjectured in [ES17, Section 6] and this has been proven in many cases (see the aforementioned reference for a list). It is known that the conjecture fails in a particular case of quiver variety of a quiver with loops, see [Tsv19, Remark 2.15]. However it is still conjectured to hold for quiver varieties of quivers without loops.

- The second condition that X has locally conical singularities is automatic if [Kal09, Conjecture 1.8] of Kaledin is true. There is so far no counterexample to the author's knowledge.

Proof. Consider the formal Poisson smoothing $\mathfrak{X}$ of X, this means we have a pullback square

$$\begin{array}{ccc} X & \hookrightarrow & \mathfrak{X} \\ \downarrow & & \downarrow \\ pt & \hookrightarrow & \Delta_t, \end{array}$$

where the top row is a map of Poisson schemes and Δ_t is the formal affine line parameterised by t. Also consider the canonical surjection $M(\mathfrak{X})[\hbar] \twoheadrightarrow \operatorname{gr}_\hbar M_\hbar(\mathfrak{X})$, let K be its kernel. We have the short exact sequence

$$0 \to K \to M(\mathfrak{X})[\hbar] \xrightarrow{(\dagger)} \operatorname{gr}_\hbar M_\hbar(\mathfrak{X}) \to 0.$$

Since the canonical map $(\dagger)$ is generically an isomorphism, the kernel K must be supported at $t = 0$. Therefore if we can show $M(\mathfrak{X})$ is a flat family over $\mathbb{C}[\![t]\!]$, then $K = 0$ and hence $M(\mathfrak{X})[\hbar] \cong \operatorname{gr} M_\hbar(\mathfrak{X})$. Quotienting by t we will get $M(X)[\hbar] \cong \operatorname{gr} M_\hbar(X)$.

It remains to show $M(\mathfrak{X})$ is a flat family over $\mathbb{C}[\![t]\!]$. Since flatness can be checked locally, we only need to check flatness of $M(\mathfrak{X}_t)$ at $t = 0$; the fibre here is just $M(X)$. Since all Poisson deformation are locally trivial at smooth points, on the smooth locus of X, the module $M(\mathfrak{X})$ is obviously flat, hence we reduce to the case of a cone.

The rest of the proof more or less follows from [PS17, Chapter 2], which we spell out the details here. By [Nam05, Theorem 18], there is a simultaneous resolution $\tilde{\mathfrak{X}}$

$$\begin{array}{ccc} \tilde{X} & \xhookrightarrow{\tilde{i}} & \tilde{\mathfrak{X}} \\ \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\tilde{\rho}} \\ X & \xhookrightarrow{i} & \mathfrak{X}. \end{array}$$

We consider two families on $\mathfrak{X}$, $\tilde{\rho}_*\Omega_{\tilde{\mathfrak{X}}_t}$ and $M(\mathfrak{X}_t)/K'$, where K' is the submodule of elements killed by powers of t.

The D-module $\tilde{\rho}_*\Omega_{\tilde{\mathfrak{X}}_t}$ is an ordinary module, that is, it is cohomologically concentrated in degree 0. Indeed, consider the short exact sequence from the universal coefficient theorem

$$0 \to H^i(\tilde{\rho}_*\Omega_{\tilde{\mathfrak{X}}_t}) \otimes_{\mathbb{C}[\![t]\!]} \mathbb{C} \to H^i(\tilde{\rho}_*\Omega_{\tilde{\mathfrak{X}}_t} \otimes^{\mathbb{L}}_{\mathbb{C}[\![t]\!]} \mathbb{C}) \to \mathrm{Tor}_1^{\mathbb{C}[\![t]\!]}(H^{i+1}(\tilde{\rho}_*\Omega_{\tilde{\mathfrak{X}}_t}), \mathbb{C}) \to 0.$$

Lemma 3.2.5. We have that

$$\tilde{\rho}_*\Omega_{\tilde{\mathfrak{X}}_t} \otimes^{\mathbb{L}}_{\mathbb{C}[\![t]\!]} \mathbb{C} \cong i_*\rho_*\Omega_{\tilde{X}},$$

which has cohomology concentrated in degree 0 since ρ is semismall.

Proof.

$$\begin{aligned} \tilde{\rho}_*\Omega_{\tilde{\mathfrak{X}}_t} \otimes^{\mathbb{L}}_{\mathbb{C}[\![t]\!]} \mathbb{C} &\cong R\tilde{\rho}_\bullet(\Omega_{\tilde{\mathfrak{X}}_t} \otimes^{\mathbb{L}}_{D_{\tilde{\mathfrak{X}}_t}} D_{\tilde{\mathfrak{X}}_t \to \mathfrak{X}_t} \otimes^{\mathbb{L}}_{\mathbb{C}[\![t]\!]} \mathbb{C}) \\ &\cong R\tilde{\rho}_\bullet(\Omega_{\tilde{\mathfrak{X}}_t} \otimes^{\mathbb{L}}_{\mathbb{C}[\![t]\!]} \mathbb{C} \otimes^{\mathbb{L}}_{D_{\tilde{\mathfrak{X}}_t}} D_{\tilde{\mathfrak{X}}_t \to \mathfrak{X}_t}) \\ &\cong R\tilde{\rho}_\bullet(\tilde{i}_*\Omega_{\tilde{X}} \otimes^{\mathbb{L}}_{D_{\tilde{\mathfrak{X}}_t}} D_{\tilde{\mathfrak{X}}_t \to \mathfrak{X}_t}) \\ &\cong \tilde{\rho}_*\tilde{i}_*\Omega_{\tilde{X}} \\ &\cong i_*\rho_*\Omega_{\tilde{X}}. \end{aligned}$$

The first equality follows from $R\tilde{\rho}_\bullet$ commutes with $\otimes^{\mathbb{L}}_{\mathbb{C}[\![t]\!]}$. The third equality follows from [Nam05, Theorem 17] that $\mathfrak{X}_t$ is locally trivial hence flat, this also implies the flatness of $\Omega_{\tilde{X}_t}$. Ask Travis if this is OK. □

Continuing with the proof of the theorem. Thus we get

- $\mathrm{Tor}_1^{\mathbb{C}[\![t]\!]}(H^i(\tilde{\rho}_*\Omega_{\tilde{\mathfrak{X}}_t}), \mathbb{C}) = 0$ for $i \neq 1$ and
- $H^i(\tilde{\rho}_*\Omega_{\tilde{\mathfrak{X}}_t}) \otimes_{\mathbb{C}[\![t]\!]} \mathbb{C} = 0$ for $i \neq 0$.

By consider the two term projective resolution of $\mathbb{C}$, the first equation says $H^i(\tilde{\rho}_*\Omega_{\tilde{\mathfrak{X}}_t})$ is t-torsion free for $i \neq 1$. Since $\tilde{\rho}$ is proper, we also know that $H^i(\tilde{\rho}_*\Omega_{\tilde{\mathfrak{X}}_t})$ is a coherent $D_{\mathfrak{X}}$-module, hence also a coherent $D_{\mathfrak{X}}[t]$-module. As for $i \neq 0$, we have that $H^i(\tilde{\rho}_*\Omega_{\tilde{\mathfrak{X}}_t})[t^{-1}] = 0$ and $H^i(\tilde{\rho}_*\Omega_{\tilde{\mathfrak{X}}_t})/tH^i(\tilde{\rho}_*\Omega_{\tilde{\mathfrak{X}}_t}) = 0$ by the second equation. If

$m_1, \dots, m_n$ are generators of $H^i(\tilde{\rho}_* \Omega_{\tilde{\mathfrak{X}}_t})$, they must be all torsion. Hence, there is an integer N such that $t^N m_1 = \cdots = t^N m_n = 0$. Therefore for $i \neq 0$, $t^N H^i(\tilde{\rho}_* \Omega_{\tilde{\mathfrak{X}}_t}) = H^i(\tilde{\rho}_* \Omega_{\tilde{\mathfrak{X}}_t}) = 0$.

Hence we know $\tilde{\rho}_* \Omega_{\tilde{\mathfrak{X}}_t}$ is concentrated in degree 0, flat and $\tilde{\rho}_* \Omega_{\tilde{\mathfrak{X}}_t} \otimes_{\mathbb{C}[\![t]\!]} \mathbb{C} = M(X)$.

Now the two flat families are isomorphic as families away from $t = 0$, and the fibres at $t = 0$ are finite length (as $\rho_* \Omega_{\tilde{X}}$ is holonomic). By [Gin86, Proposition 1.1.2], the two fibres $M(X)/K''$ where K'' is the image of the canonical map $K'/tK' \to M(\mathfrak{X})/tM(\mathfrak{X}) = M(X)$ and $\rho_* \Omega_{\tilde{X}}$ must be isomorphic in the Grothendieck group of holonomic D-modules on X. By our assumption, $M(X) \cong \rho_* \Omega_{\tilde{X}}$, therefore K'' must be zero. But, since $\mathrm{Tor}_1^{\mathbb{C}[\![t]\!]}(M(\mathfrak{X})/K', \mathbb{C}) = 0$, the map $K'/tK' \to K''$ is an isomorphism, so $K'/tK' = 0$ as well. Since $K'[t^{-1}] = 0$, we again get that $K' = 0$ as in the case of $H^i(\tilde{\rho}_* \Omega_{\tilde{\mathfrak{X}}_t})$ for $i \neq 0$. Thus $M(\mathfrak{X})$ is torsion-free and hence flat over $\mathbb{C}[\![t]\!]$.

To get that $M_\hbar(X) \to M(X)[\![\hbar]\!] \cong \rho_* \Omega_{\tilde{X}}[\![\hbar]\!]$ is a filtered isomorphism, we apply Lemma 3.1.14 and the fact that $M_\hbar(X)$ is complete. □

By applying the pushforward π_*, we easily see that:

Corollary 3.2.6. Let X and $\mathcal{O}_\hbar(X)$ satisfy the conditions of the theorem above, then $\mathbf{HP}^{dR}_\bullet(X)[\![\hbar]\!] \cong \mathbf{HH}^{dR}_\bullet(\mathcal{O}_\hbar(X)) \cong H^{\dim X - \bullet}(\tilde{X}, \mathbb{C}[\![\hbar]\!])$.

Combined with Lemma 3.1.3, we get that:

Corollary 3.2.7. If X and $\mathcal{O}_\hbar(X)$ satisfy the conditions of the theorem above, then there exists a short exact sequence of $\mathbb{C}[\![\hbar]\!]$-modules

$$0 \to H^{\dim X}(\tilde{X}, \mathbb{C}[\![\hbar]\!]) \to \mathbf{HH}_0(\mathcal{O}_\hbar(X)) \to \mathcal{O}_X \to 0,$$

where $\hbar$ acts trivially on $\mathcal{O}(X)$. And $\mathbf{HH}_0(\mathcal{O}_\hbar(X)[\hbar^{-1}]) \cong H^{\dim X}(\tilde{X}, \mathbb{C}(\!(\hbar)\!))$.

Proof. The first equation follows from $\frac{\mathcal{O}_\hbar(X)/[\mathcal{O}_\hbar(X), \mathcal{O}_\hbar(X)]}{\hbar \mathcal{O}_\hbar(X)/[\mathcal{O}_\hbar(X), \mathcal{O}_\hbar(X)]} \cong \frac{\mathcal{O}_\hbar(X)}{\hbar \mathcal{O}_\hbar(X)} \cong \mathcal{O}(X)$ as $\mathbb{C}[\![\hbar]\!]$-modules. By definition the numerator is $\mathbf{HH}_0(\mathcal{O}_\hbar(X))$ and by Lemma 3.1.3 the denominator is $\mathbf{HH}_0^{dR}(\mathcal{O}_\hbar(X))$, which by the previous corollary is isomorphic to $H^{\dim X}(\tilde{X}, \mathbb{C}[\![\hbar]\!])$. As $\hbar$ is acting on $\mathcal{O}(X)$ trivially, $\mathcal{O}(X)[\hbar^{-1}] = 0$. □

Remark 3.2.8. One can also upgrade the conjecture of Etingof–Schedler to claim that $M_\hbar(X)[\hbar^{-1}] \cong \rho_* \Omega_{\tilde{X}}(\!(\hbar)\!)$ for a symplectic resolution $\rho : \tilde{X} \to X$ and that $\mathbf{HH}_i^{dR}(\mathcal{O}_\hbar(X)) \cong H^{\dim X - i}(\tilde{X}, \mathbb{C}(\!(\hbar)\!))$, for every quantisation.

3.3 Holonomicity

It is known that if X has finitely many symplectic leaves then $M(X)$ is holonomic as a D_X-module [ES09, Theorem 1.1]. We also get a similar result for $M_\hbar(X)$. To shorten the notation, by $M_\hbar(X)/\hbar^n$ we mean $M_\hbar(X)/\hbar^n M_\hbar(X)$.

Proposition 3.3.1. When X has finitely many symplectic leaves, $M_\hbar(X)/\hbar^n$ is a holonomic D_X-module for all n.

Proof. From the canonical surjection (†), we get that

$$M(X)[\hbar]/\hbar^n \twoheadrightarrow (\operatorname{gr}_\hbar M_\hbar(X))/\hbar^n$$

is a surjective homomorphism of D-modules. Since surjection preserves holonomicity and $(\operatorname{gr}_\hbar M_\hbar(X))/\hbar^n \cong \operatorname{gr}_\hbar(M_\hbar(X)/\hbar^n)$, we see that $\operatorname{gr}_\hbar(M_\hbar(X)/\hbar^n)$ is a holonomic D-module. Since holonomic modules are closed under extension (see [HTT08, Proposition 3.1.2.]), $M_\hbar(X)/\hbar^n$ is also a holonomic D-module.

□

It is known that $\mathbf{HH}_0(\mathcal{O}_\hbar(X)[\hbar^{-1}])$ is finite dimensional over $\mathbb{C}(\!(\hbar)\!)$ and $\mathcal{O}_\hbar(X)[\hbar^{-1}]$ has finitely many finite dimensional representations when X has finitely many symplectic leaves by [ES09, Corollary 3.13]. We give a slightly different proof and generalise to Hochschild–de Rham homology.

Theorem 3.3.2. If X has finitely many symplectic leaves then $\mathbf{HH}^{dR}_\bullet(\mathcal{O}_\hbar(X))$ is finitely-generated over $\mathbb{C}[\![\hbar]\!]$. In particular $\mathbf{HH}_0(\mathcal{O}_\hbar(X)[\hbar^{-1}])$ is finite dimensional over $\mathbb{C}(\!(\hbar)\!)$ and $\mathcal{O}_\hbar(X)[\hbar^{-1}]$ has finitely many finite dimensional representations.

Proof. We know that

$$\operatorname{gr}_i M_\hbar(X) = \hbar^i M_\hbar(X)/\hbar^{i+1} M_\hbar(X) \twoheadrightarrow \operatorname{gr}_{i+1} M_\hbar(X) = \hbar^{i+1} M_\hbar(X)/\hbar^{i+2}.$$

Since each $\operatorname{gr}_i M_\hbar(X)$ is holonomic (hence has finite length), there is an integer N such that $\operatorname{gr}_N M_\hbar(X) \cong \operatorname{gr}_{N+i} M_\hbar(X)$ for all $i > 0$. Thus $\pi_*(\operatorname{gr} M_\hbar(X))$ is finitely-generated over $\mathbb{C}[\hbar]$ by $\{\pi_* \operatorname{gr}_i M_\hbar(X)\}_{i \le N}$. Since there is a spectral sequence $\pi_*(\operatorname{gr} M_\hbar(X)) \implies \operatorname{gr}(\pi_* M_\hbar(X))$, $\operatorname{gr} \mathbf{HH}^{dR}_\bullet(\mathcal{O}_\hbar(X))$ must also be finitely-generated over $\mathbb{C}[\hbar]$. It is a general result that if A is complete and $\operatorname{gr} M$ is finitely generated as a $\operatorname{gr} A$-module, then M is finitely generated as an A-module (see [AM69, Prop 10.24]). This proves the result. □

3.4 A conjecture on Kontsevich's quantisation

Now assume further that X is smooth affine. The Kontsevich Formality Theorem says that there is a L_∞ quasi-isomorphism

$$T_{\text{poly}} \xrightarrow{L_\infty} D_{\text{poly}},$$

where $T_{\text{poly}} := (\bigwedge^\bullet_{\mathcal{O}(X)} T^1(X))[1]$ is the dgla of (shifted) polyvector fields on X and $D_{\text{poly}} := C^\bullet(\mathcal{O}(X))[1]$ is the dgla of (shifted) Hochschild chains on X (computing Hochschild cohomology of $\mathcal{O}(X)$). The Poisson structure π is an MC element on the left hand side, we can form the MC twisting and get an L_∞ quasi-isomorphism

$$(T_{\text{poly}}, d_\pi) \xrightarrow{L_\infty} (D_{\text{poly}}, d_{\text{Hoch}_\hbar}),$$

where $d_{\text{Hoch}_\hbar} = d_{\text{Hoch}} + [\mu_\hbar - \mu, \cdot]$, $\mu_\hbar$ is the Kontsevich quantisation. It follows that

$$\mathbf{HP}^\bullet(\mathcal{O}(X)(\!(\hbar)\!), \pi_\hbar) \xrightarrow{\sim} \mathbf{HH}^\bullet(\mathcal{O}(X)(\!(\hbar)\!), \star).$$

See [Bel+16, Chapter II, Section 4.11] and the references therein.

A similar version involving homology is also true.

We conjecture that a similar statement involving Poisson-de Rham and Hochschild–de Rham homology is true:

Conjecture 3.4.1. When $\mathcal{O}_\hbar(X)$ is the Kontsevich quantisation, then

$$\mathbf{HP}^{dR}_\bullet(X)[\hbar^{-1}] \cong \mathbf{HH}^{dR}_\bullet(\mathcal{O}_\hbar(X))[\hbar^{-1}].$$

Recall one version of the Kontsevich formality theorem says that there is a L_∞ quasi-isomorphism

$$(\Omega_{\text{poly}}, \mathcal{L}_\pi) \xrightarrow{L_\infty} D_{\text{poly}},$$

where $\Omega_{\text{poly}} := (\bigwedge^\bullet_{\mathcal{O}(X)} \Omega^1(X))[1]$ is the dgla of (shifted) differential forms on X and $D_{\text{poly}} := C^\bullet(\mathcal{O}_\hbar(X))[1]$ is the dgla of (shifted) Hochschild chains on $\mathcal{O}_\hbar(X)$ (computing Hochschild homology of $\mathcal{O}_\hbar(X)$).

It is tempting to write down a 'proof' of this conjecture by considering tensoring this quasi-isomorphism with $D_X[\![\hbar]\!]$, and get that

$$\Omega_{\text{poly}} \otimes_{\mathcal{O}(X)[\![\hbar]\!]} D_X[\![\hbar]\!] \xrightarrow{\sim} D_{\text{poly}} \otimes_{\mathcal{O}(X)[\![\hbar]\!]} D_X[\![\hbar]\!].$$

Taking H^0 of both sides, we get $M(X)[\![\hbar]\!] \xrightarrow{\sim} M_\hbar(X)$ and the conjecture follows. However, this proof is wrong as the maps d_π and $d_{\text{Hoch}_\hbar}$ are not $\mathcal{O}_X$-linear hence we won't get a complex of D-modules.

Chapter 4

Quantum topology and skein theory

4.1 Two possible approaches

Recall from the introduction we fix $X := \mathrm{Hom}(\pi_1(T^2), SL_n)///SL_n \cong T^{n-1}/S_n$ to be the SL_n-character variety of the 2-torus T. Let $\mathbb{T}$ be a maximal torus of SL_n and $W = S_n$ be its Weyl group. We have the algebra of W-invariant $D_q(\mathbb{T})^W$ of the $(n-1)$-quantum torus. Let $\mathcal{O}_\hbar(X) := D_q(\mathbb{T})^W$, this is a quantisation of the character variety. We wish to compute the zeroth Hochschild homology of this algebra.

There are two ways of looking at this, one way is by the localisation theorem of McGerty–Nevins [MN11], which says:

Theorem 4.1.1. If A is a quantisation of $\mathcal{O}(X)$, $\rho : \tilde{X} \to X$ is a symplectic resolution and $\mathcal{A}$ is a quantisation of $\mathcal{O}(\tilde{X})$ lifting A (*i.e.*, that $\Gamma(\tilde{X}, \mathcal{A}) = A$), moreover A has finite global dimension, then there is a derived Morita equivalence $D^b(A\text{-mod}) \cong D^b(\mathcal{A}\text{-mod})$.

As Hochschild homology is a Morita invariant ([Ric91], [Kel98]), we can look at $\mathbf{HH}_i(\mathcal{A})$. Analogously to Nest–Tsygan, this is $H^{\dim X - i}(\tilde{X})$ as $\tilde{X}$ is smooth symplectic. In particular, $\mathbf{HH}_0(\mathcal{A})$ is given by the top cohomology. However, we are not sure if the quantisation will lift to the resolution.

Or we can look at the Hochschild–de Rham picture from the previous chapter:

Theorem 4.1.2. Let $\rho : \tilde{X} \twoheadrightarrow X$ be a projective symplectic resolution such that

- $\rho_* \Omega_{\tilde{X}} \cong M(X)$,

- X has locally conical singularities.

Assume further that the quantisation $A = \mathcal{O}_\hbar(X)$ extends to a quantisation $\mathcal{O}_\hbar(\mathfrak{X})$ on a (one-parameter) smoothing $\mathfrak{X}$ of X. Then $\mathbf{HH}_0(\mathcal{O}_\hbar(X)[\hbar^{-1}]) \cong H^{\dim X}(\tilde{X}, \mathbb{C}(\!(\hbar)\!))$.

Since our $X = T^{n-1}/S_n$ is a finite quotient singularity by [BS21a, Proposition 2.8], we know that formally locally X has conical singularity by for example [BS21a, Theorem A.1] and [Her68]. The conjecture of Etingof–Schedler holds in this case because formally locally the singularity is a product of the singularities for the Hilbert–Chow resolution $\mathrm{Hilb}^n \mathbb{C}^2 \to \mathrm{Sym}^n \mathbb{C}^2$ and affine spaces (see later), for which we know the conjecture holds, see [ES17, Chapter 7]. Finally recall from the introduction, the extended quantisation on the smoothing is the spherical double affine Hecke algebra (see [BBJ18, Section 1.5]).

4.2 Hilbert scheme

Regardless of the approach we take, we need to compute the top cohomology of the resolution.

Let $T^n := (\mathbb{C}^*)^n$ be the n-torus. We are interested in finding the resolution $\mathrm{Hilb}_0^n(T)$ of T^{n-1}/S_n, where T^{n-1} embeds into T^n via the map $i : T^{n-1} \to T^n$ that sends

$$((a_1, b_1), \dots, (a_{n-1}, b_{n-1}))$$

to

$$((a_1, b_1), \dots, (a_{n-1}, b_{n-1}), (a_1^{-1} \dots a_{n-1}^{-1}, b_1^{-1} \dots b_{n-1}^{-1})).$$

This can be thought as a multiplicative version of the reflection representation of S_n.

Let C_n be the cyclic group of order n. There is a $C_n \times C_n$-covering map

$$f : T^{n-1}/S_n \times T \to T^n/S_n$$

that sends

$$(\{\{(a_1, b_1), \dots, (a_{n-1}, b_{n-1})\}\}, (c, d))$$

to

$$\{\{(a_1 c, b_1 d), \dots, (a_{n-1} c, b_{n-1} d), (a_1^{-1} \dots a_{n-1}^{-1} c, b_1^{-1} \dots b_{n-1}^{-1} d)\}\}.$$

Here we are using the multi-set notation $\{\{\ \}\}$.

The $C_n \times C_n$ action amounts to the choice of the n-th roots of $a_1 \dots a_n$ and $b_1 \dots b_n$. The action is given by

$$(x,y)\cdot(\{\{(a_1,b_1),\dots,(a_{n-1},b_{n-1})\}\},(c,d))$$
$$=(\{\{(xa_1,yb_1),\dots,(xa_{n-1},yb_{n-1})\}\},(x^{-1}c,y^{-1}d)).$$

Moreover, f respects this action, with trivial action on T^n/S_n. This makes f a $C_n \times C_n$ equivariant map.

Recall that there is a symplectic resolution $\rho : \mathrm{Hilb}^n(T) \to T^n/S_n$ (see [FN03] and [Fu05, Example 2.4]). As the map f is a covering, the resolution lifts as symplectic resolution respects (étale) base change (see [BS21b, Lemma 5.2]). Furthermore, the map $\bar{i} : T^{n-1}/S_n \to T^n/S_n$ induced from $i : T^{n-1} \to T^n$ factors through f. In summary, we have the following diagram:

$$\begin{array}{ccccc}
\mathrm{Hilb}^n_0(T) & \longrightarrow & \widehat{\mathrm{Hilb}^n(T)} & \xrightarrow{\tilde{f}} & \mathrm{Hilb}^n(T) \\
\downarrow & & \downarrow{\scriptstyle \tilde{\rho}} & & \downarrow{\scriptstyle \rho} \\
T^{n-1}/S_n & \longrightarrow & T^{n-1}/S_n \times T & \xrightarrow[f]{} & T^n/S_n,
\end{array}$$

where the composition of the bottom row is $\bar{i}$, $\mathrm{Hilb}^n_0(T)$ is the pullback of the outer square and is a symplectic resolution of T^{n-1}/S_n (see [BS21a, page 11]) and

$$\widehat{\mathrm{Hilb}^n(T)} := \mathrm{Hilb}^n(T) \times_{T^n/S_n} (T^{n-1}/S_n \times T)$$

is the pullback of the right square and it is a symplectic resolution of $T^{n-1}/S_n \times T$.

Moreover the map $\tilde{f}$ is also a $C_n \times C_n$ covering. As the left square is also a pullback square, we have:

Lemma 4.2.1.
$$\widehat{\mathrm{Hilb}^n(T)} = \mathrm{Hilb}^n_0(T) \times T.$$

The cohomology of $\widehat{\mathrm{Hilb}^n(T)}$ is related to the cohomology of the resolution $\mathrm{Hilb}^n_0(T)$ of T^{n-1}/S_n by the Künneth formula. Let h_i be the (complex) betti number of $\mathrm{Hilb}^n_0(T)$. Then

$$\dim H^{2n}(\widehat{\mathrm{Hilb}^n(T)}) = h_{2n-2},$$
$$\dim H^{2n-1}(\widehat{\mathrm{Hilb}^n(T)}) = 2h_{2n-2} + h_{2n-3},$$
$$\dim H^{i}(\widehat{\mathrm{Hilb}^n(T)}) = h_i + 2h_{i-1} + h_{i-2} \text{ for } 2 \le i \le 2n-2,$$
$$\dim H^{1}(\widehat{\mathrm{Hilb}^n(T)}) = h_1 + 2$$

One can prove by induction that

$$h_i = \sum_{j=0}^{i} (-1)^j (j+1) \dim H^{i-j}(\widehat{\mathrm{Hilb}^n(T)}).$$

Therefore we just need to calculate $\dim H^i(\widehat{\mathrm{Hilb}^n(T)})$.

In the paper [Nie07, Corollary 3] (proved by generalising Nakajima's result), local systems L_χ^ν on T^2 were introduced so that we can calculate the cohomology using the following isomorphism:

$$\bigoplus_{n\geq 0} H^*(\widehat{\mathrm{Hilb}^n(T)}, \mathbb{C}[2n]) \cong \bigoplus_{\chi\in(C_n\times C_n)^\vee} \mathrm{Sym}(\bigoplus_{\nu\geq 1} H^*(T, L_\chi^\nu[2])) \tag{4.1}$$

There is an isomorphism of bi-graded vector spaces, where the first grading is by the cohomology degree, and the second grading (called weighting) is the number of points n on the left and $H^*(T, L_\chi^\nu)$ has weight ν. Note that the Sym on the right hand side of the isomorphism is understood to be the graded symmetric power in cohomology degree.

From this, we can compute all cohomologies of $\widehat{\mathrm{Hilb}^n(T)}$. Note that

$$\begin{aligned} H^*(T, L_\chi^\nu[2]) &\cong H^*(S^1\times S^1, L_{\chi_1}^\nu[1]\boxtimes L_{\chi_2}^\nu[1]) \\ &\cong H^*(S^1, L_{\chi_1}^\nu[1]) \otimes H^*(S^1, L_{\chi_2}^\nu[1]) \\ &\cong \delta_{\chi_1^\nu,1}\delta_{\chi_2^\nu,1}(\mathbb{C}[0]\oplus\mathbb{C}[-1])^{\otimes 2}\{\nu\}. \end{aligned}$$

Here we are using $[-]$ for (cohomological) grading and $\{-\}$ for weighting.

Identifying $(C_n\times C_n)^\vee \cong \mathbb{Z}/n\mathbb{Z}\times\mathbb{Z}/n\mathbb{Z}$ and rewriting $\chi = (\chi_1,\chi_2)$ as (a,b), the right hand side of equation 4.1 becomes

$$\bigoplus_{(a,b)\in(\mathbb{Z}/n\mathbb{Z}\times\mathbb{Z}/n\mathbb{Z})} \mathrm{Sym}(\bigoplus_{\nu\geq 1}\delta_{a\nu,0}\delta_{b\nu,0}(\mathbb{C}[0]\oplus\mathbb{C}[-1])^{\otimes 2}\{\nu\}).$$

We can simplify this using elementary number theory:

$$\begin{aligned}\delta_{a\nu,0}\delta_{b\nu,0} \text{ is non-zero if and only if } & n|a\nu \text{ and } n|b\nu,\\ & \text{if and only if } \frac{n}{\gcd(a,n)}|\nu \text{ and } \frac{n}{\gcd(b,n)}|\nu,\\ & \text{if and only if } \frac{n}{\gcd(a,b,n)}|\nu.\end{aligned}$$

Here we are taking $\gcd(0,n) = n$ and $\gcd(0,a,n) = \gcd(a,n)$.

So the right hand side of equation 4.1 becomes

$$\bigoplus_{(a,b)\in(\mathbb{Z}/n\mathbb{Z}\times\mathbb{Z}/n\mathbb{Z})} \mathrm{Sym}(\bigoplus_{k\geq 1}(\mathbb{C}[0]\oplus\mathbb{C}^2[-1]\oplus\mathbb{C}[-2])\{k\frac{n}{\gcd(a,b,n)}\}).$$

For the cohomology in top degree, we have:

$$\bigoplus_{n\geq 0} H^{2n}(\widehat{\mathrm{Hilb}^n(T)},\mathbb{C}) \cong \bigoplus_{(a,b)\in(\mathbb{Z}/n\mathbb{Z}\times\mathbb{Z}/n\mathbb{Z})} \mathrm{Sym}(\bigoplus_{k\geq 1}\mathbb{C}\{k\frac{n}{\gcd(a,b,n)}\}) \tag{4.2}$$

Let $\mathcal{P}(n)$ be the number of partitions of n. Recall the k-th Jordan's totient function $J_k : \mathbb{N} \to \mathbb{N}$ is defined as

$$J_k(n) = n^k \prod_{p|n}\left(1 - \frac{1}{p^k}\right).$$

This number $J_k(n)$ equals the number of k-tuples of positive integers that are less than or equal to n and that together with n form a coprime set of $k+1$ integers. When $k = 1$, this is the usual Euler's totient function. See [SC04, Section 3.7.1].

Furthermore, If $f, g : \mathbb{N} \to \mathbb{C}$ are two arithmetic functions from the positive integers to the complex numbers, the Dirichlet convolution $f \star g$ is a new arithmetic function defined by:

$$(f\star g)(n) \ = \ \sum_{d\,|\,n} f(d)\,g\left(\frac{n}{d}\right) \ = \ \sum_{ab\,=\,n} f(a)\,g(b),$$

where the sum extends over all positive divisors d of n, or equivalently over all distinct pairs (a, b) of positive integers whose product is n. See [SC04, Section 2.2.1].

Taking weight n of equation 4.2, we get that

Theorem 4.2.2.

$$
\begin{aligned}
\dim H^{2n}(\widehat{\mathrm{Hilb}^n(T)}) &= \sum_{(a,b)\in(\mathbb{Z}/n\mathbb{Z}\times\mathbb{Z}/n\mathbb{Z})} \mathcal{P}(\gcd(a,b,n)) \\
&= \sum_{d|n} \mathcal{P}(d) J_2(\frac{n}{d}) \\
&= \mathcal{P}\star J_2(n),
\end{aligned}
$$

where J_2 is the second Jordan's totient function.

Note this has a natural $(\mathbb{Z}/n\mathbb{Z}\times\mathbb{Z}/n\mathbb{Z})$-grading.

For $n=p$ a prime, this number is $\mathcal{P}(p)+p^2-1$. When $p=2$, this gives $2+4-1=5$.

Recall from the introduction that Sam Gunningham, David Jordan and Monica Vazirani showed that

$$
\mathrm{SkAlg}_{SL(n)}(T^2) \cong \mathbf{HH}_0(D_q(\mathbb{T})^W), \tag{4.3}
$$

and

$$
\mathrm{Sk}_{SL_n}(T^3) \cong \mathbf{HH}_0(D_q(\mathbb{T})^W) \bigoplus \mathbb{C}^k. \tag{4.4}
$$

It immediately follows from Theorem 4.1.2 and the above theorem:

Corollary 4.2.3.

$$
\dim \mathrm{SkAlg}_{SL(n)}(T^2) = \mathcal{P}\star J_2(n).
$$

Recall further that $\mathrm{Sk}_{SL(n)}(T^3)$ has a $H^1(T^3,\mathbb{Z}/n\mathbb{Z}) = (\mathbb{Z}/n\mathbb{Z})^3$ grading, and there is a mapping class group $\mathrm{MCG}(T^3) = SL(3,\mathbb{Z})$ action on $\mathrm{Sk}_{SL_n}(T^3)$, compatible with the grading via the quotient morphism $SL(3,\mathbb{Z}) \to SL(3,\mathbb{Z}/n\mathbb{Z})$.

Hence $\mathrm{Sk}_{SL(n)}(T^3)_{(a,b,0)} = \mathrm{Sk}_{SL(n)}(T^3)_{g\cdot(a,b,0)}$, for any $g \in SL(3,\mathbb{Z}/n\mathbb{Z})$.

We know that:

$$
\begin{aligned}
\dim \mathrm{Sk}_{SL(n)}(T^3)_{(a,b,0)} &= \dim \mathrm{SkAlg}_{SL(n)}(T^2)_{(a,b)} \\
&= \dim \mathbf{HH}_0(D_q(\mathbb{T})^W)_{(a,b)} \\
&= \mathcal{P}(\gcd(a,b,n)),
\end{aligned}
$$

where the first equality follows from isomorphism 4.4, the second equality follows from isomorphism 4.3 and the last equality follows from the above corollary. We need Theorem 4.1.2 to be an iso under group action.

An arbitrary element of $(\mathbb{Z}/n\mathbb{Z})^3$ is a translate by $SL_3(\mathbb{Z}/n\mathbb{Z})$ of a weight of form

$(d,0,0)$, for some divisor d of n. Moreover, the Jordan totient function J_3 (resp. J_2) arise very naturally: $J_3(n/d)$ (resp. $J_2(n/d)$) is precisely the cardinality of the orbit of $(d,0,0)$ (resp. $(d,0)$). See [NH05, Theorem 4.9].

We deduce the following beautiful equation:

Corollary 4.2.4.

$$\begin{aligned}\dim \mathrm{Sk}_{SL(n)}(T^3) &= \sum_{d|n} \dim \mathrm{SkAlg}_{SL_n}(T^2)_{(d,0)} J_3(\frac{n}{d}) \\ &= \sum_{d|n} \mathcal{P}(\gcd(d,0,n)) J_3(\frac{n}{d}) \\ &= \sum_{d|n} \mathcal{P}(d) J_3(\frac{n}{d}) \\ &= \mathcal{P} \star J_3(n).\end{aligned}$$

4.3 Higher cohomology

Let $P_r(m)$ be the *set* of r-component multipartitions of m, that is, an r-tuple of partitions $\lambda(1),\ldots,\lambda(r)$, where each $\lambda(i)$ is a partition of some a_i and the a_i sum to m. Let $\mathcal{P}_r(m)$ be the size of this set.

Define the subset $P_4^{(0,1,1,2)}(m,i)$ inside of $P_4(m)$ consisting elements of $(\lambda(1),\lambda(2),\lambda(3),\lambda(4))$ such that $l(\lambda(2))+l(\lambda(3))+2l(\lambda(4))=i$. Let $\mathcal{P}_4^{(0,1,1,2)}(m,i)$ be the *size* of this subset.

To take account of graded symmetric power, we want for $i=2,3$, those partitions $\lambda(i)$ of a_i such that consecutive parts of $\lambda(i)$ are different. This defines $\mathcal{P}_{4,\mathrm{gr}}^{(0,1,1,2)}(m,i)$.

The generating polynomial for each n is a multiple of $(1+2t+t^2)$, and the quotient is the generating polynomial for $h_i = \dim H^i(\widetilde{T^{n-1}/S_n})$.

Theorem 4.3.1. In general, for the $(2n-i)$th cohomology, $\dim H^{2n-i}(\widehat{\mathrm{Hilb}^n(T)})$ has the following description:

$$\dim H^{2n-i}(\widehat{\mathrm{Hilb}^n(T)}) = \sum_{(a,b)\in(\mathbb{Z}/n\mathbb{Z}\times\mathbb{Z}/n\mathbb{Z})} \mathcal{P}_{4,\mathrm{gr}}^{(0,1,1,2)}(\gcd(a,b,n),i).$$

Proof. We consider elements of the following direct sum

$$\bigoplus_{(a,b)\in(\mathbb{Z}/n\mathbb{Z}\times\mathbb{Z}/n\mathbb{Z})} \mathrm{Sym}(\bigoplus_{k\geq 1}(\mathbb{C}_x[0]\oplus\mathbb{C}^2_{y,z}[-1]\oplus\mathbb{C}_w[-2])\{k\frac{n}{\gcd(a,b,n)}\}).$$

For fixed (a, b), an element of the direct sum will have the form

$$x_1^{\alpha_1} x_2^{\alpha_2} \dots x_{k_1}^{\alpha_{k_1}} \dots w_1^{\delta_1} w_2^{\delta_2} \dots w_{k_4}^{\delta_{k_4}}$$

such that

$$\beta_1 + \dots + \beta_{k_2} + \gamma_1 + \dots + \gamma_{k_3} + 2(\delta_1 + \dots + \delta_{k_4}) = 2n - i$$

and

$$\alpha_1 + 2\alpha_2 \dots + k_1\alpha_{k_1} + \beta_1 + \dots + k_2\beta_{k_2} + \gamma_1 + \dots + k_3\gamma_{k_3} + \delta_1 + \dots + k_4\delta_{k_4} = \gcd(a, b, n),$$

where $\beta_i, \gamma_i = 0$ or 1 for all i. This gives the desired set. □

Theorem 4.3.2. For large n, the list $\dim H^i(\widehat{\mathrm{Hilb}^n(T)})$ (for $0 \leq i \leq 2n$) stabilises to the number of bi-partitions of i wherein odd parts are distinct (and even parts are unrestricted).

This sequence is given by OEIS:A273225. Therefore, for 'large' n, the dimensions of cohomology of $\dim H^i(\widehat{\mathrm{Hilb}^n(T)})$ is given by: 1, 2, 3, 6, 11, 18, 28, 44, 69, 104...

We give two proofs, one uses an explicit bijection and the other one uses generating function. The second proof will generalise to Hilbert scheme of points of other smooth surfaces, see the remark below.

Proof. First we show when n is large, there are no contributions from a, b unless when $a = b = 0$, in which case $\gcd(a, b, n) = n$. Suppose $a \neq 0 \neq b$, then $\gcd(a, b, n) \leq n/2$. Therefore

$$2n - i + l(\lambda_2) + l(\lambda_3) = 2l(\lambda_2) + 2l(\lambda_3) + 2l(\lambda_4) \leq 2(a + b + c + d) \leq n.$$

Hence $l(\lambda_2) + l(\lambda_3) \leq i - n$, which is impossible as soon as $i < n$. Therefore we can assume $a = b = 0$.

Next we show that there are always $n - i$ parts of 1's in λ_4. (Note this step is not strictly necessary.) Assume there are $n - j$ parts of 1's, we want to show $j \leq i$. Let the length of λ_4 be $n - j + k$, then $d \geq 2k + n - j$, hence $a + b + d \leq j - 2k$. Then

$$2n - i = l(\lambda_2) + l(\lambda_3) + 2l(\lambda_4) \leq a + b + d + 2d \leq j - 2k + 2(n - j + k),$$

and $j \leq i$.

We can delete the $n-i$ parts of 1's in λ_4 and hence the counting problem has become:

4-partition $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ of i with λ_2, λ_3 distinct and

$$l(\lambda_2) + l(\lambda_3) + 2l(\lambda_4) = i.$$

We want to show this is the number of bi-partitions of i wherein odd parts are distinct.

Let $\lambda_1 = (1^{\alpha_i}, \dots, k_1^{\alpha_{k_1}}), \dots, \lambda_4 = (1^{\delta_i}, \dots, k_4^{\delta_{k_4}})$. Create a new bi-partition:

$$\mu_1 = (1^{\beta_1}, 2^{\alpha_1}, 3^{\beta_2}, 4^{\alpha_2}, \dots),$$
$$\mu_2 = (1^{\gamma_1}, 2^{\delta_2}, 3^{\gamma_2}, 4^{\delta_3}, \dots).$$

This has odd parts distinct and it is a bi-partition of

$$2a + 2b - l(\lambda_2) + 2c + 2d - l(\lambda_3) - 2l(\lambda_4) = 2i - i = i.$$

Note that this process is clearly invertible.

□

We begin the other proof by writing down the generating function for both sides.

The generating function for the number of partitions of n such that the odd parts are distinct is

$$\prod_{k\geq 1} \frac{1+t^{2k+1}}{1-t^{2k}},$$

therefore the generating function for the number of bi-partitions of n such that the odd parts are distinct is:

$$\prod_{k\geq 1} \frac{(1+t^{2k+1})^2}{(1-t^{2k})^2}.$$

The generating function for $H^i(\mathrm{Hilb}^n(T))$ is:

$$\prod_{k\geq 1} \frac{(1+t^{2k-1}x^k)^2}{(1-t^{2k-2}x^k)(1-t^{2k}x^k)} = \sum_n f_n(t)x^n,$$

where $f_n(t) = \sum_i \beta_i(\mathrm{Hilb}^n(T))t^i$ is the Poincaré series for $\mathrm{Hilb}^n(T)$ (see [CM00, Theorem 5.2.1]).

The left hand side is

$$\frac{1}{1-x} \prod_{k\geq 1} \frac{(1+t^{2k-1}x^k)^2}{(1-t^{2k}x^{k+1})(1-t^{2k}x^k)},$$

with the coefficients of the infinite product are still positive. Writing the infinite product as $\sum_{q\geq 0} g_q(t)x^q$. We have the equality:

$$\sum_{p\geq 0} x^p \cdot \sum_{q\geq 0} g_q(t)x^q = \sum_{n} f_n(t)x^n.$$

Collecting coefficients of x^n, we have:

$$f_n(t) = \sum_{q=0}^{n} g_q(t).$$

Therefore as n goes to infinity,

$$\begin{aligned} f_{n\to\infty}(t) &= \prod_{k\geq 1} \frac{(1+t^{2k-1}x^k)^2}{(1-t^{2k}x^{k+1})(1-t^{2k}x^k)}\Bigg|_{x=1} \\ &= \prod_{k\geq 1} \frac{(1+t^{2k-1})^2}{(1-t^{2k})^2}, \end{aligned}$$

which is what we wanted.

Remark 4.3.3. The generating function approach allows us to generalise the convergence of cohomology of Hilbert scheme of points on any smooth surface X. Recall from [CM00, Theorem 5.2.1] that the generating function for $H^i(\mathrm{Hilb}^n(X))$ is given by

$$\sum_{n\geq 0}\sum_{i} \dim H^i(\mathrm{Hilb}^n(X))t^i x^n = \prod_{k\geq 1} \frac{(1+t^{2k-1}x^k)^{\beta_1}(1+t^{2k+1}x^k)^{\beta_3}}{(1-t^{2k-2}x^k)^{\beta_0}(1-t^{2k}x^k)^{\beta_2}(1-t^{2k+2}x^k)^{\beta_4}},$$

where β_i is the i-th betti number of X. Using the same trick as above, we can show that for a connected surface, the cohomology converges and has generating function:

$$\prod_{k\geq 1} \frac{(1+t^{2k-1})^{\beta_1}(1+t^{2k+1})^{\beta_3}}{(1-t^{2k})^{\beta_2+1}(1-t^{2k+2})^{\beta_4}}.$$

The coefficients of t^i have the following combinatorial description: the number of coloured partitions of i such that

- there are β_1 colours for length 1 parts,
- $1+\beta_2$ colours for length 2 parts,
- $\beta_1+\beta_3$ colours for odd length ≥ 3 parts,
- $1+\beta_2+\beta_4$ colours for even length ≥ 4 parts,
- odd parts have distinct length.

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